MATHEMATICS

10

Tamil Nadu Textbook Society

MATHEMATICS

STANDARD X

1. Untouchability is a sin
2. Untouchability is a crime
3. Untouchability is inhuman.

Tamil Nadu Textbook Society
MADRAS

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The scientist is a lover of truth for the very love of truth itself, wherever it may lead.

Luther Burbank

1. SET LANGUAGE

1-1. Sets : Definition

Let us recall what we have learnt about 'Sets' in the previous standards.

A set is a collection of well-defined objects.

'A tall man', 'A good student' : These are not well-defined. So these are not Sets.

'Persons who are more than 160 cm in height'

'Students who have scored more than 85 marks in all subjects' — These are well-defined. So these are sets.

If the set $A = \{5, 8, 9, 15\}$,

then, $5 \in A$, $9 \in A$, $10 \notin A$, $12 \notin A$

Set Notation :

1. Roster Method :

   { 1, 2, 3, 4 }, { Teacher, Pupils, Parents }, { Rose, Jasmine }

2. Descriptive method :

   { Natural numbers }, { Flowers }, { Pupils in a class }

3. Set builder notation :

   $\{x/x \text{ is an even number}\}$, $\{x/1 < x < 10, x \in N\}$

The number of elements in a set is called the cardinal number of the set.

eg., $n\{1, 2, 5, 9\} = 4$

If two sets have the same elements, they are called Equal sets.

eg., { 1, 2, 3 } { 3, 1, 2 } these are equal sets.

If two sets have the same cardinal numbers, they are called 'Equivalent Sets'

eg., { 1, 2, 3 } { *a*, *b*, *c* } are equivalent sets, since each set has the cardinal number 3.

If the cardinal number of a set is infinity, that set is called an Infinite Set.

eg., $N = \{ 1, 2, 3, \ldots \}$

If the cardinal number of a set is finite, the set is called a finite set.

eg., { 7, 8, 9, 12, 17 }

If the cardinal number of a set is zero, the set is called the Null Set or the Empty Set.

eg., { }

We denote the null set by the alphabet ϕ

'The set of two-digit numbers less than 10',

'The set of hexagons with 4 sides',

'The set of integers between 4 and 5' are some examples of the empty set.

Subsets :

If every element of a set B is an element of a set A, then B is called a **Subset** of A. This is denoted by $B \subseteq A$. By definition, we find that every set is a subset of itself. The null set is considered to be a subset of all sets.

If $A = \{ 5, 8, 12, 17, 25 \}$,

then { 5, 8, 12 }, { 8, 17, 25 }, {12, 25 }, { 8 }, { } are some of the subsets of A.

If the set contains at least one element that is not in a subset, then the subset is called a **proper subset** of the set.

{ 2 }, { 2, 3 } are proper subsets of { 1, 2, 3 }.

A set is an improper subset of itself.

A set which has all the subsets of a set as its elements is called the power set of the set.

i. e. The set of all the subsets of a set is the power set of the set. The power set of A is denoted by $P(A)$.

Example :

If $A = \{ 1, 2 \}$,

$P(A) = \{ \{ 1, 2 \}, \{ 1 \}, \{ 2 \}, \{ \ \} \}$

$n[P(A)] = 4 = 2^2$

If $X = \{1, 2, 3\}$,

$P(X) = \{ \{ 1, 2, 3 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 2, 3 \},$
$\{ 1 \}, \{ 2 \}, \{ 3 \} \{ \ \} \}$

$n[P(X)] = 8 = 2^3$

In general, if $n(A) = p$, then $n[P(A)] = 2^p$

Exercise 1-1.

1. Write the following in Set-builder notation :

(a) { 1, 3, 5, 7,... } (b) { 3, 6, 9,... }
(c) { 4, 5, 6, 7, 8, 9 } (d) { −1, −2, −3,... }
(e) { Triangle, Quadrilateral, Pentagon,... }

2. Match the sets and their subsets :

Sets		Subsets	
(a)	{ 4, 7, 9, 11, 12 }	(i)	$\{x/x = 5n + 1, n \in N\}$
(b)	{ −4, −2, −1, 7, 8, 11 }	(ii)	{ 7, 11, 4 }
(c)	{ 1, 2, 3, ... }	(iii)	{ 8, 16, 24, ... }
(d)	{ 2, 4, 6, ... }	(iv)	{ − 1 }

3. Write the power sets of the following sets.

(a) { 5, 9 } (b) { a, b, c } (c) { 9 }

4. The cardinal numbers of some sets are given below. Find the cardinal numbers of their power sets.

(a) 4 (b) 6 (c) 2 (d) 3 (e) 1

5. (a) If $n[P(A)] = 32$, $n(A) = ?$

(b) If $n[P(A)] = 512$, $n(A) = ?$

(c) If $n(A) = k$, $n[P(A)] = ?$

(d) If $n(A) = 0$, $n[P(A)] = ?$

ANSWERS

1. (a) $\{ x/x$ is an odd integer $\}$

(b) $\{ x/x$ is a multiple of 3, $x \in N \}$

(c) $\{ x/3 < x < 10, x \in N)$

(d) $\{x/x$ is a negative integer $\}$

(e) $\{ x/x$ is a polygon $\}$

2. (a) ii (b) iv (c) i (d) iii

3. (a) $\left\{ \{5, 9\}, \{5\}, \{9\}, \{\ \} \right\}$

(b) $\left\{ \{a. b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \{\ \} \right\}$

(c) $\left\{ \{9\}, \{\ \} \right\}$

4. (a) 16 (b) 64 (c) 4 (d) 8 (e) 2

5. (a) 5 (b) 9 (c) 2^k (d) 1,

for $A = \{\ \}$; $P(A) = \left\{ \{\ \} \right\}$

1-2. Set Operations

Let us recall

If $A = \{1, 3, 5, 7\}$; $B = \{2, 3, 4, 5, 8\}$

then $A \cap B = \{3, 5\}$

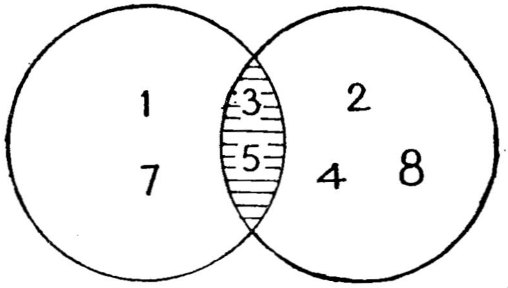
$A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$

$A \cap B$ means the intersection of two sets A and B. It consists of those elements that are both in A and in B.

$$A \cap B = \{ x / x \in A \text{ and } x \in B \}$$

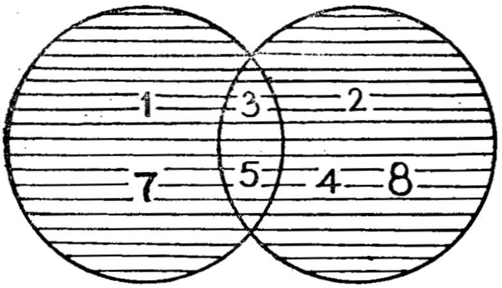
$A \cup B$ means the union of two sets A and B.

It consists of those elements either in A or in B or in both A and B.



$A \cap B$

Fig. 1-1



$A \cup B$

Fig. 1-2

If the intersection of two sets is the null set, then the sets are called disjoint sets.

If the intersection of two sets is not the null set, then the sets are called intersecting sets or overlapping sets.

Example :

If $A = \{ 2, 4, 5, 7, 9\}$; $B = \{ 3, 6, 9, 12, 15, 18 \}$, then find $A \cup B$ and $A \cap B$.

Find $n(A)$, $n(B)$, $n(A \cup B)$, $n(A \cap B)$ and find the relation among them.

$A \cup B = \{ 2, 3, 4, 5, 6, 7, 9, 12, 15, 18 \}$

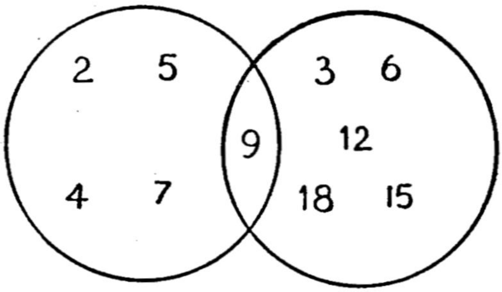


Fig. 1-3

$A \cap B = \{ 9 \}$

$n(A) = 5$ $\qquad$ $n(B) = 6$

$n(A \cup B) = 10$ $\qquad$ $n(A \cap B) = 1$

$n(A) + n(B) = n(A \cup B) + n(A \cap B) = 11$

With the help of some more sets, verify that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

If $A \cap B = \phi$. $n(A \cup B) = n(A) + n(B)$

Example :

Find $A \cup B$ and $A \cap B$

1. $A = \{ 4, 7, 9, 11, 13 \}$, $B = \{ 2, 3, 4, 11, 13, 15 \}$

   $A \cap B = \{ 4, 11, 13 \}$

   $A \cup B = \{ 2, 3, 4, 7, 9, 11, 13, 15 \}$

2. $A = \{$ odd Natural numbers $\}$

   $B = \{$ even Natural numbers $\}$

   $A = \{ 1, 3, 5, 7, \ldots \}$, $B = \{ 2, 4. 6, 8, \ldots \}$

   $A \cap B = \phi$, A and B are disjoint sets.

   $A \cup B = \{ 1, 2, 3, 4, \ldots \} = \{$ Natural numbers $\}$.

Exercise 1-2.

1. $A = \{0, 1, 3, 4\}$, $B = \{1, 2, 3, 5\}$
   $C = \{1, 3, 4, 5\}$ Find $A \cup B$, $B \cup C$ and $A \cup C$

2. $A = \{5, 7, 3, 9\}$, $B = \{3, 7, 4, 6\}$
   $C = \{7, 4, 6, 10\}$ Find $A \cap B$, $B \cap C$ and $A \cap C$.

3. $A = \{3, 7, 9, 11\}$ $B = \{4, 5, 6, 8\}$; Find $A \cup B$ and $A \cap B$.

4. Fill up the blanks.

	$n(A)$	$n(B)$	$n(A \cap B)$	$n(A \cup B)$
(a)	5	4	3	—
(b)	6	7	—	8
(c)	3	5	—	6
(d)	4	4	—	5
(e)	6	—	2	7
(f)	3	—	0	5

5. $A = \{1, 2, 3, 4, 5\}$, $A \cup B = 1, 2, 3, \ldots, 9\}$,
   $A \cap B = \{4, 5\}$ Find the set B.

6. Draw a Venn diagram and verify the following:
   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$

ANSWERS

1. $\{0, 1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 5\}$, $\{0, 1, 3, 4, 5\}$
2. $\{3, 7\}$, $\{4, 6, 7\}$, $\{7\}$
3. $\{3, 4, 5, 6, 7, 8, 9, 11\}$, $\{\ \}$
4. $\{6, 5, 2, 3, 3, 2\}$
5. $\{4, 5, 6, 7, 8, 9\}$

1-3. Set Identity 1 :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Let us take the sets

$A = \{4, 5, 11, 13\}$ $B = \{6, 8, 11, 12, 13\}$ and

$C = \{3, 4, 8, 12, 13\}$

$B \cap C = \{8, 12, 13\}$

$\therefore A \cup (B \cap C) = \{4, 5, 11, 13\} \cup \{8, 12, 13\}$
$= \{4, 5, 8, 11, 12, 13\}$

$A \cup B = \{4, 5, 6, 8, 11, 12, 13\}$

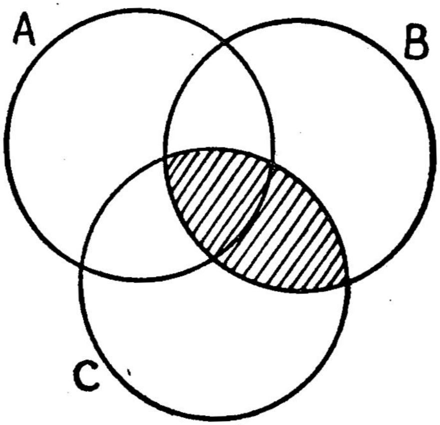
$A \cup C = \{3, 4, 5, 8, 11, 12, 13\}$

$(A \cup B) \cap (A \cup C) = \{4, 5, 6, 8, 11, 12, 13\} \cap \{3, 4, 5, 8, 11, 12, 13\}$
$= \{4, 5, 8, 11, 12, 13\}$

From these we find $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

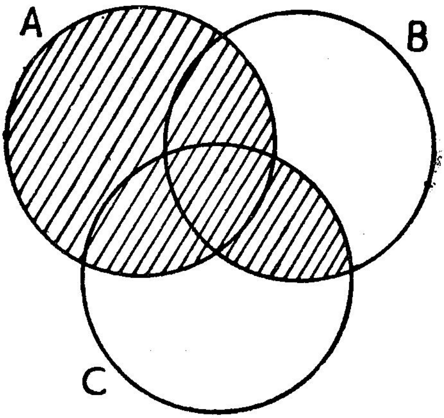
Take some more sets and verify the above statement.

We can illustrate the fact through Venn diagrams.



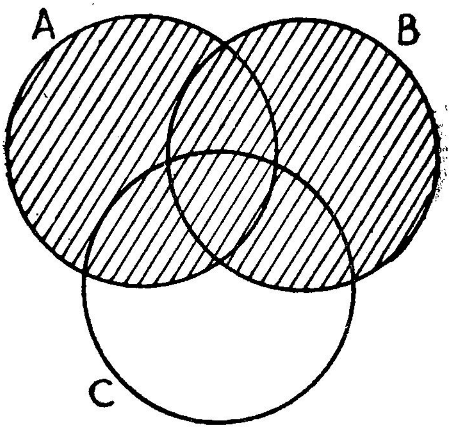
$B \cap C$

Fig. 1-4



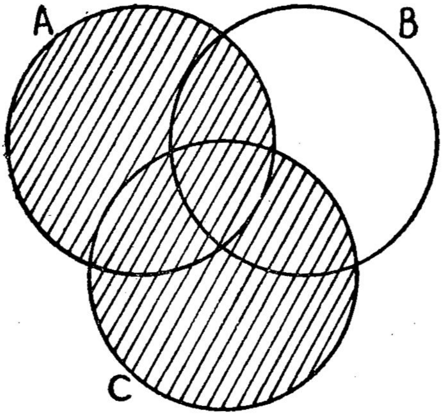
$A \cup (B \cap C)$

Fig. 1-5



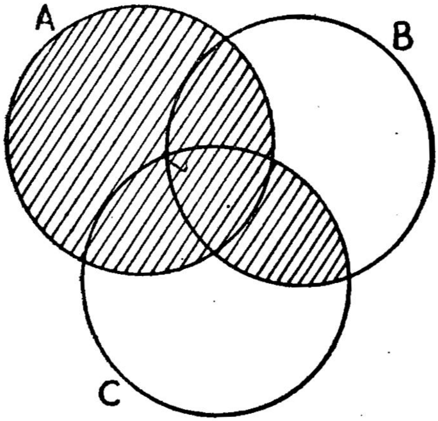
$A \cup B$

Fig. 1-5



$A \cup C$

Fig. 1·7



$(A \cup B) \cap (A \cup C)$

Fig. 1-8

Exercise 1-3

1. Find $A \cup B$, $A \cup C$ and $B \cap C$ in the following examples and verify the identity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

	Set A	Set B	Set C
1.	$\{0, 1, 2, 3, 4, 5\}$	$\{1, 3, 5, 7\}$	$\{2, 4, 7, 9, 11\}$
2.	$\{a, b, c, d, e\}$	$\{b, d, e\}$	$\{a, e, f, g\}$
3.	$\{2, 5, 8, 9\}$	$\{3, 4, 5, 6, 9, 11\}$	$\{2, 8, 9\}$
4.	$\{1, 2, 3\}$	$\{3, 4, 5\}$	$\{6, 7, 8, 9\}$
5.	$\{p, q, r, s\}$	$\{2, 3, 5\}$	$\{2, 5, p, r\}$

ANSWERS

1. $\{0, 1, 2, 3, 4, 5, 7\}$; $\{0, 1, 2, 3, 4, 5, 7, 9, 11\}$; $\{7\}$; $\{0, 1, 2, 3, 4, 5, 7\}$

2. $\{a, b, c, d, e\}$; $\{a, b, c, d, e, f, g\}$; $\{e\}$; $\{a, b, c, d, e\}$

3. $\{2, 3, 4, 5, 6, 8, 9, 11\}$; $\{2, 5, 8, 9\}$; $\{9\}$; $\{2, 5, 8, 9\}$;

4. $\{1, 2, 3, 4, 5\}$; $\{1, 2, 3, 6, 7, 8, 9\}$; $\{\ \phi\ \}$; $\{1, 2, 3\}$;

5. $\{p, q, r, s, 2, 3, 5\}$; $\{p, q, r, s, 2, 5\}$; $\{2, 5\}$; $\{p, q, r, s, 2, 5\}$.

1·4 Set Identity 2 :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Let us take three sets, $A = \{1, 3, 4, 5, 7\}$

$B = \{2, 3, 4, 7, 8, 9\}$ and

$C = \{4, 5, 8, 9\}$

$$B \cup C = \{2, 3, 4, 5, 7, 8, 9\}$$

$$A \cap (B \cup C) = \{1, 3, 4, 5, 7\} \cap \{2, 3, 4, 5, 7, 8, 9\}$$
$$= \{3, 4, 5, 7\}$$
$$A \cap B = \{3, 4, 7\}$$
$$A \cap C = \{4, 5\}$$
$$(A \cap B) \cup (A \cap C) = \{3, 4, 7\} \cup \{4, 5\}$$
$$= \{3, 4, 5, 7\}$$
$$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Exercise - 1-4

I. Take the sets A, B and C given in Exercise 1·3 and find $A \cap B$, $A \cap C$, $B \cup C$. Verify the identity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

II. Verify the identity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ with the help of a Venn diagram.

ANSWERS

1. { 1, 3, 5 }, {2, 4} , {1, 2, 3, 4, 5, 7, 9, 11 } { 1, 2, 3, 4, 5 }
2. { b, d, e }, { a, e } , {a b, d, e, f, g} , {a, b, d, e}
3. { 5, 9 }, {2, 8, 9} , {2, 3, 4, 5, 6, 8, 9, 11} {2, 5, 8, 9}
4. { 3 }, { }, { 3, 4, 5, 6, 7, 8, 9 }, { 3 }
5. { }, { p, r }, { p, r, 2, 3, 5, }, {p, r}

1·5 Complementary Sets

Let us recall :

If $\mathcal{E}$ = {5, 6, 7, 8, 9} is the Universal set,
and A = {7, 8}
then A' = {5, 6, 9} is the complement of A.

A', the complement of A, is the set containing all the elements in the Universal set, but not in A.

Exercise 1-5

1. $\mathcal{E}$ = {1, 2, 3, 4, 5, ...} , A = {2, 4, 6 8, ...} Find A
2. $\mathcal{E}$ = {a, b, c, d, e, f, g. h} , A = {b, d. e. f} , B = {a, b, e, f, h} ; Find A', B', $A' \cup B'$ and $A' \cap B'$

3. $\mathcal{E} = \{ 11, 13, 14, 15, 17, 19, 21, 23 \}$,
$A = \{ 13, 15, 17 \}$; $B = \{ 15, 17, 21, 23 \}$
$C = \{ 13, 17, 19, 21 \}$
Find $(A \cup B)'$, $(B \cup C)'$ and $(A \cup C)'$.

4. Taking $\mathcal{E}$ as the Universal set, find $A \cup A'$, $A \cap A'$ and ϕ

5. $\mathcal{E} = \{ 1, 2, 3, 4, \ldots, 25 \}$
If $A = \{$ multiples of 3 less than 25 $\}$, Find A'

6. (a) $\mathcal{E} = \{$ Natural numbers $\}$,
$A = \{$ Odd numbers $\}$, Find A',

(b) $\mathcal{E} = \{$ integers $\}$,
$B = \{$ Positive integers $\}$ Find B'.

(c) $\mathcal{E} = \{$ Real numbers $\}$,
$X = \{$ irrational numbers $\}$ Find X'.

(d) $\mathcal{E} = \{$ English alphabets $\}$
$V = \{$ English vowels $\}$ Find V'.

7. Find A' and B' in the first and the third questions given above and then find out $(A')'$ and $(B')'$. Find their relation with A and B.

8. Find $A \cup A'$, $A \cap A'$ and ϕ' in the previous sum. Show them in a Venn diagram.

9. (a) $n(\mathcal{E}) = 20$ $n(A) = 12$ $n(A') = \ldots\ldots\ldots$

(b) $n(\mathcal{E}) = 42$ $n(X') = 26$ $n(X) = \ldots\ldots\ldots$

(c) $n(\mathcal{E}) = 30$ $n(M') = 16$ $n(M) = \ldots\ldots\ldots$

(d) $n(\mathcal{E}) = 40$ $n(A') = 6$ $n(B) = 10$. If $B \subset A$, find the number of elements of A which are not in B.

10. (a) $n(\mathcal{E}) = 145$; $n(A \cup B) = 98$; $n[(A \cup B)'] = ?$

(b) $n(\mathcal{E}) = 84$; $n(A) = 58$; $n(B) = 54$;
$n[(A \cup B)'] = 12$; $n(A \cap B) = ?$

ANSWERS

1. { 1, 3, 5, 7... }

3. { 11, 14, 19 } { 11, 14 } { 11, 14, 15, 19, 21, 23 }

5. { 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23}.

6. (a) {even natural numbers} ,
   (b) {0, negative integers}
   (c) { rational numbers }
   (d) { English consonants }

7. $(A')' = A$; $(B')' = B$.

9. (a) 8, (b) 16, (c) 14, (d) 24.

10 (a) 47, (b) $n(A \cup B) = 72$;
   $n(A \cap B) = 58 + 54 - 72 = 40$.

1·6 De Morgan's Law (1): (a) $(A \cup B)' = A' \cap B'$

Let $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$

then $A' = \{4, 5, 6, 7, 8\}$ $B' = \{1, 2, 7, 8\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$(A \cup B)' = \{1, 2, 3, 4, 5, 6\}' = \{7, 8\}$

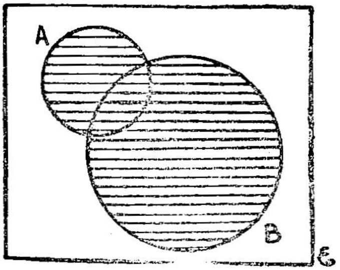
$A' \cap B' = \{4, 5, 6, 7, 8\} \cap \{1, 2, 7, 8\} = \{7, 8\}$

$\therefore$ $(A \cup B)' = A' \cap B''$.

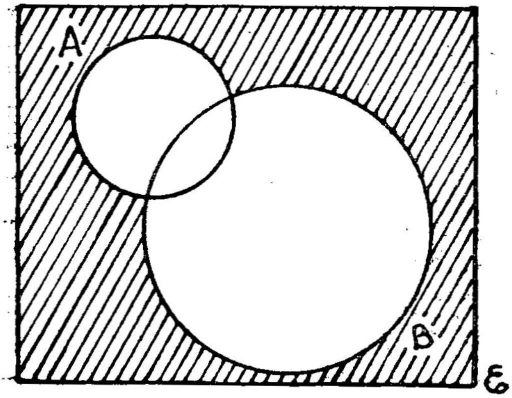
The complement of the union of two sets is the intersection of their complements.

This is one of the De Morgan's laws.

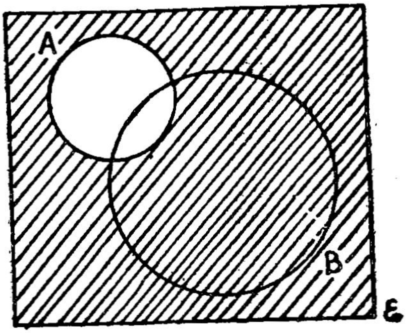
We can verify the law with the help of a Venn diagram.



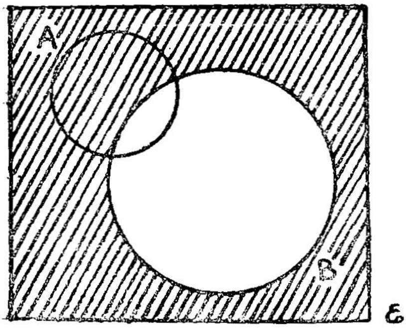
$A \cup B$
Fig. 1-9.



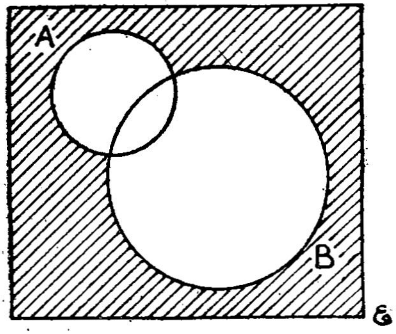
$(A \cup B)'$
Fig. 1-10.



A'
Fig 1-11.



B'
Fig 1-12



$A' \cap B'$

Fig. 1-13.

Exercise 1-6

Verify De Morgan's first law with the help of the following sets :

	Universal set	set A	set B
1.	$\{1, 2, 3, ..., 10\}$	$\{1, 3, 5, 7\}$	$\{2, 5, 7, 9, 10\}$
2.	$\{x \mid 6<x<18, x \in N\}$	$\{8, 12, 16\}$	$\{6, 12, 14\}$
3.	$\{x \mid -4<x<6, x \in Z\}$	$\{x \mid 0<x<4, x \in Z)\}$	$\{x \mid -2<x<3, x \in Z\}$
4.	$\{2, 5, 7, 9, 11, 13, 14\}$	$\{9, 11, 13\}$	$\{2, 5, 9, 14\}$
5.	$\{5, 6, 9, 11, 13, 17, 18\}$	$\{6, 9, 13, 17\}$	$\{5, 9, 17, 18\}$

1-7. De Morgan's Law 2 : $(A \cap B)' = A' \cup B'$

Let us have $\mathcal{E} = \{1, 2, 3, 4, ... 10\}$

$A = \{3, 5, 7, 8, 9\}$

and $B = \{2, 5, 6, 7\}$

$A' = \{1, 2, 4, 6, 10\}$, $B' = \{1, 3, 4, 8, 9, 10\}$

$A \cap B = \{5, 7\}$

$(A \cap B)' = \{1, 2, 3, 4, 6, 8, 9, 10\}$

$(A' \cup B') = \{1, 2, 4. 6, 10\} \cup \{1, 3, 4, 8, 9, 10\}$

$= \{1, 2, 3, 4, 6, 8, 9, 10\}$

$\therefore \quad (A \cap B)' = A' \cup B'$.

The complement of the intersection of two sets is the union of the complements of the sets. This is the second of the De Morgan's laws.

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Note well the places where the symbols of union and intersection ($\cup$ and $\cap$) occure and remember De Morgan's laws.

Exercise 1-7

1. Using the sets in Ex. 1-6, verify the 2nd law of De Morgan.
2. Verify $(A \cup B)' = A' \cap B'$, using a Venn diagram.

1-8. Solutions Through Venn Diagrams

Recall : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example 1 :

Out of 100 pupils who sat for the examination, 48 passed in Mathematics, 40 in English and 25 passed in both the subjects. How many pupils have failed in both ?

Solution :

Let A = { Those who passed in Mathematics }

$n(A) = 48$

Let B = { Those who passed in English }

$n(B) = 40$

$A \cap B$ = { Those who passed both in Maths and English }

$n(A \cap B) = 25$

Note down the details in a Venn diagram.

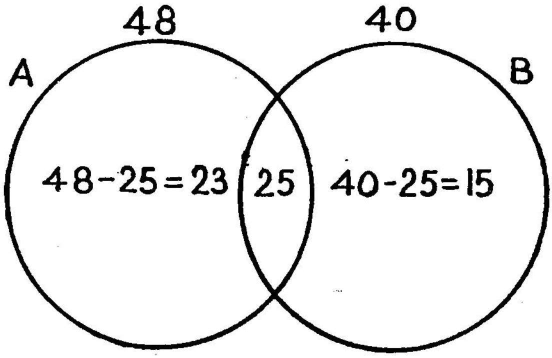


Fig. 1-14.

From the Venn diagram,

$n(A \cup B) = 23 + 25 + 15 = 63$

Hence, the number of pupils who have failed in both the subjects

$= 100 - 63 = 37.$

Exercise 1-8

Find the solutions through Venn diagrams :

(Mental Sums 1 6)

1. $n(A) = 20$, $n(B) = 50$, $n(A \cap B) = 15$, $n(A \cup B) = ?$
2. $n(A) = 72$, $n(B) = 80$, $n(A \cap B) = 40$, $n(A \cup B) = ?$
3. $n(A) = 35$, $n(B) = 50$, $n(A \cup B) = 70$, $n(A \cup B) = ?$
4. $n(A) = 52$, $n(B) = 40$, $n(A \cup B) = 80$, $n(A \cup B) = ?$
5. $n(A \cup B) = 60$, $n(B) = 30$, $n(A \cap B) = 20$, $n(A) = ?$
6. $n(A \cup B) = 60$, $n(A) = 25$, $n(A \cap B) = 10$, $n(B) = ?$
7. Out of 200 persons living in a street, 120 drink coffee, 70 drink tea. 60 persons drink both coffee and tea. Find the number of persons who drink neither coffee nor tea.

8. In a class, 26 students have taken Physics, 21 students have taken Chemistry and 11 have taken both Physics and Chemistry. If there is no student who has not taken either Physics or Chemistry, find (a) the number of students who have taken only Physics (b) the number who have taken only Chemistry and (c) the total number of students in the class.

9. In a factory 40 persons work in the production department and 30 persons in the office. 20 persons work both in the production department and in the office. Find the total number of persons working in the factory.

10. 32 students of a class went to Coimbatore on an excursion. 20 students visited the Agricultural University and 21 visited the exhibition in the Forest College. Find the number of those who did not see the exhibition in the Forest College.

ANSWERS

(7) 70; (8) 15, 10, 36; (9) 50; (10) 11.

1-9. Problems Involving three sets

Formula : If $A \cap B \cap C \neq \phi$,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cup B) - n(B \cup C) - n(A \cup C) + n(A \cap B \cap C)$$

It is easier to find the solution with the help of a Venn diagram than with the formula.

Example 1 :

There was a celebration for three days in a school. 140 persons attended the function on the first day, 100 on the second day and 170 on the third day. 35 persons attended on all the three days, 55 on the first two days, 80 on the first and the third day and 65 persons on the second and the third day. Find how many attended (a) only on the first day (b) only on the second day (c) only on the third day.

Solution :

Let A = { Those who attended on the first day }

B = { Those who attended on the second day }

and C = { Those who attended on the third day }

$n(A) = 140$, $n(B) = 100$; $n(C) = 170$.

n { Those who attended on all the three days }

$= 35 = n(A \cap B \cap C)$

n {Those who attended on the first two days}

$= 55 = n(A \cap B)$

n {Those who attended on the 2nd and 3rd days}

$= 65 = n(B \cap C)$

n {Those who attended on the first and the third day} $= 80 = n(A \cap C)$

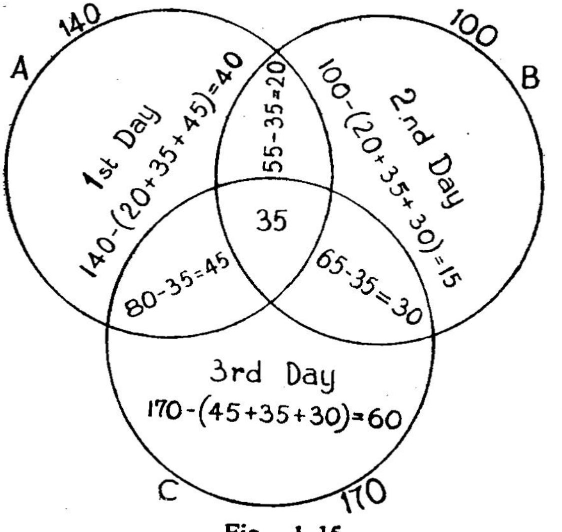


Fig. 1-15

It is known from the figure,

Number of persons who attended only on the first day = 40

Number of persons who attended only on the 2nd day = 15

Number of persons who attended only on the 3rd day = 60

Exercise 1-9

Solve with the help of Venn diagrams.

1. There are 60 pupils in a class. Out of them 25 play football, 20 play hockey and 30 play cricket. 12 play football and hockey. 9 play hockey and cricket. 13 play cricket and football. 5 play all the three games. Find how many of them play

   (a) none of the games

   (b) only cricket and

   (c) only football

2. In the 10th Std. examination in a school, 50 pupils passed in English, 70 in Mathematics and 60 in Science. 15 pupils passed in both English and Mathematics; 20 in both Mathematics and Science and 30 passed in both English and Science. 8 pupils passed in all the three subjects. Find how many passed (a) in English only (b) in Mathematics only and (c) in Science only.

3. The following are the particulars collected in a survey of listeners by an All India Radio Station :

Item	*Listeners*
Music	= 60%
Talks	= 40%
Talks only	= 10%
Drama only	= 10%
Those who listened to Music and Talks but not Drama	= 15%
Those who listened to Music and Drama but not Talks	= 20%
Those who listened to all the three items	= 5%

From these particulars find the percentage of

(a) those who listened to music only

(b) those who listened to none of these items

(c) Which item was most listened to ?

4. Particulars of participation in events in sports are given below :

Events	No. of participants
100 m	30
200 m	18
400 m	12

Out of these, 3 athletes participated in all the three items, 18 in 100 m only and 3 in 200 m only. None of those who took part in 100 m, participated in 400 m. Find the total number of athletes who participated in the sports.

5. $n(A) = 30$, $n(B) = 20$, $n(C) = 40$.

$n(A \cap B) = 13$, $n(B \cap C) = 11$, $n(A \cap C) = 18$

$n(A \cap B \cap C) = 6$; Find $n(A \cup B \cup C)$.

6.

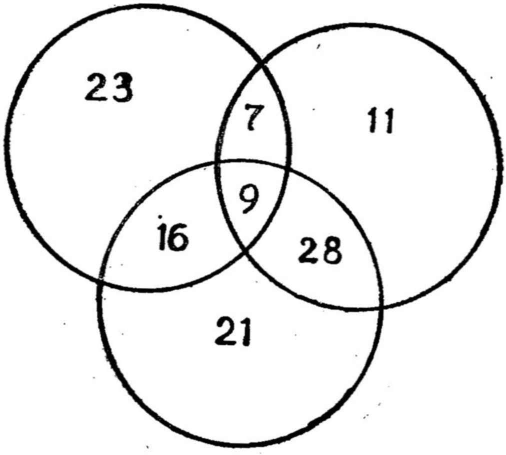


Fig. 1-16

From the Venn diagram, take away one element. Construct a problem using the remaining particulars. You can have a number of problems, by removing various elements one by one. Solve them.

1-10. Function - Definition

Let us recall what we learnt about functions in the previous standard.

Let A and B be non-empty sets.

If every element of A is associated with one and only one element in B, the relation is called a '**Function**'.

(Note carefully the words 'Every' and 'one and only one')

The element 'b' of set B associated with the element 'a' of set A is called the image of 'a'.

'a' associated with 'b' is called the pre-image of 'b'.

In the following figures, the elements of $A = \{a_1, a_2, a_3, a_4\}$ are associated with b_1, b_2, b_3 in four different ways.

(a)

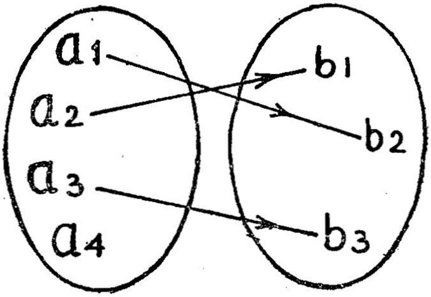


Fig. 1-17

(b)

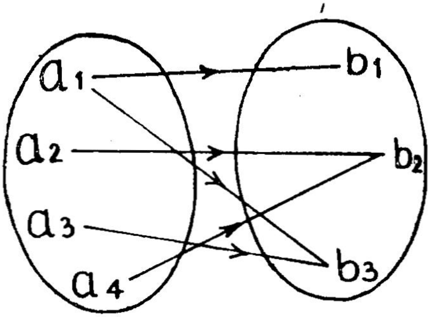


Fig. 1-18

(c)

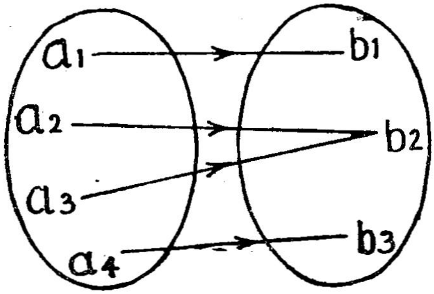


Fig. 1-19

(d)

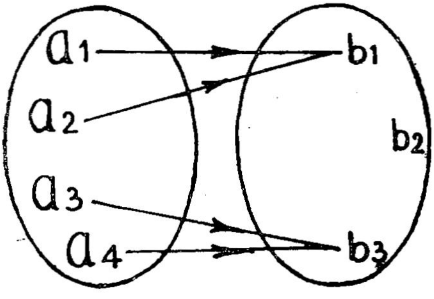


Fig. 1-20

Figure	Domain	Range	Set of relation	Whether a function or not
(a)	$\{a_1, a_2, a_3, a_4\}$	$\{b_1, b_2, b_3\}$	$\{(a_1, b_2), (a_2, b_1), (a_3, b_3)\}$	Not a function. a_4 has no image. 'Every member'- condition is not satisfied.
(b)	$\{a_1, a_2, a_3, a_4\}$	$\{b_1, b_2, b_3\}$	$\{(a_1, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_3), (a_4, b_2)\}$	Not a function. 'a_1' has two images. 'One and only one' condition is not satisfied
(c)	$\{a_1, a_2, a_3, a_4\}$	$\{b_1, b_2, b_3\}$	$\{(a_1, b_1), (a_2, b_2), (a_3, b_2), (a_4, b_3)\}$	Function
(d)	$\{a_1, a_2, a_3, a_4\}$	$\{b_1, b_2, b_3\}$	$\{(a_1, b_1), (a_2, b_1), (a_3, b_3), (a_4, b_3)\}$	Function

Note : 1. In (a) and (b), the number of elements in the sets of relation are 3 and 5 respectively. The number of elements in the domain is 4. As there must be one and only one image for each element, if the number of (elements) ordered pairs in the set of relation is $>$ or $<$ 4, then the relation cannot be a function.

If it is equal to 4, it may or may not be a function.

2. If the ordered pairs in the set of relation have every member of the domain as the first member exactly once, then the relation is a function.

Example :

1. $\{(7, 3), (-2, 5), (-3, 4), (-1, 4)\}$

The first members of the ordered pairs are different.

$\therefore$ this is a function.

Domain $= \{7, -2, -3, -1\}$; Range $= \{3, 5, 4\}$

2. $A = \{1, 3, 5\}$ $B = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$

Relation set $f = \left\{(1, 1), \left(3, \frac{1}{3}\right), \left(5, \frac{1}{5}\right)\right\}$

Range $= \left\{1, \frac{1}{3}, \frac{1}{5}\right\}$. This is a function,

3. $A = \{-3, -2, -1, 0, 1, 2, 3\}$; $B = N$

The relation is $f(x) = x + 2$

Co-domain $= \{1, 2, 3 ...\}$

$f(-3) = -3 + 2 = -1.$ $f(-2) = -2 + 2 = 0$

-1, 0 are not in the co-domain. So this is not a function.

Exercise 1-10

1. In the following figures some relations are shown. Find whether they are functions.

Find the domain, co-domain, range and the relation set of each function.

(a)

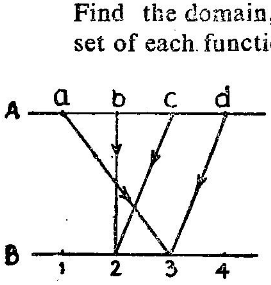


Fig. 1-21.

(b)

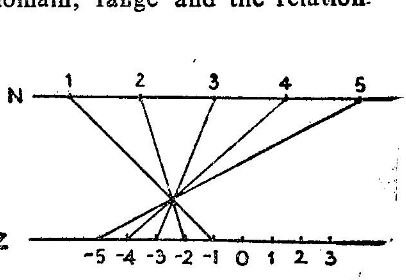
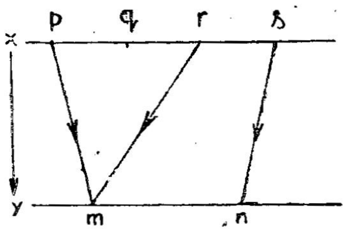


Fig. 1-22.

(c)



Ftg. 1-23.

(d)

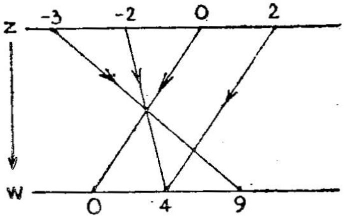


Fig. 1-24.

2. $A = \{0, 1, 2, 3, 4\}$, $B = \{-4, -3, -2, -1, 0\}$ relation $f : A \rightarrow B$, such that $f(x) = -2x$. Check whether it is a function.

3. $\{(3, 8), (-3, 2), (1, 6), (0, 5), (-5, 0)\}$. Is this relation set a function ?

4. Relation f : $A \rightarrow B$. The rule is $f(x) = x^2 - 4$. If $A = B = N$, verify whether 'f' is a function.

5. Relation f : $Z \rightarrow Z$. The rule is

$$f(x) = 2x + 1 \text{ if } x > 0$$
$$= -x \text{ if } x < 0$$

Show that the relation is a function. Find the range for the domain $\{-5, -2, -1, 0, 1, 3\}$

ANSWERS

1. (a) Function ; $\{a, b, c, d\}$, $\{1, 2, 3, 4\}$, $\{2, 3\}$
$\{(a, 3), (b, 2) (c, 2) (d, 3)\}$

(b) Function ; $\{1, 2, 3, 4, 5\}$;
$\{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$
$\{-5, -4, -3, -2, -1\}$
$\{(1,-1), (2,-2), (3,-3), (4,-4), (5,-5)\}$

(c) Not a function. There is no image for 'q'.

(d) Function $\{-3, -2, 0, 2\}$, $\{0, 4, 9\}$ $\{0, 4, 9\}$
$\{(-3, 9), (-2, 4), (0,10), (2, 4)\}$

2. This is not a function. There is no image for 3 and 4.

3. Yes, it is a function. $f(x) = x + 5$.

4. Not a function. 1 and 2 have no immage.

5. $\{5, 2, 1, 3, 7\}$

1-11. Function-Evalution

Let us take the set of ordered pairs,

$f: \{(0, 1), (3, 5), (4, -3)\}$

The domain of 'f' is $\{0, 3, 4\}$

The range of 'f' is $\{1, 5, -3\}$

$f(0) = 1$, $f(3) = 5$, $f(4) = -3$

If an ordered pair (x, y) is an element of the function 'f' then $f(x) = y$.

Example 1 :

If $f = \{(x, y) / y = 4x - 7\}$, find $f(0)$, $f(2)$ and $f(5)$.

Solution :

$$\begin{aligned} f(x) &= 4x-7 \\ f(0) &= 4(0)-7 \\ &= 0-7 \\ &= -7 \end{aligned}$$

$$\begin{aligned} f(x) &= 4x-7 \\ f(2) &= 4(2)-7 \\ &= 8-7 \\ &= 1. \end{aligned}$$

$$\begin{aligned} f(x) &= 4x-7 \\ f(-5) &= 4(-5)-7 \\ &= -20-7 \\ &= -27. \end{aligned}$$

Example 2 :

In the function $f(x) = 3x^2 - x + 2$, the domain of 'f' is $\{1, 3, -3\}$. Find the Range of 'f'.

Solution :

$$\begin{aligned} f(x) &= 3x^2 - x + 2 \\ f(1) &= 3(1)^2 - 1 + 2 \\ &= 3 - 1 + 2 \\ &= 4. \end{aligned}$$

$$\begin{aligned} f(x) &= 3x^2 - x + 2 \\ f(3) &= 3(3)^2 - 3 + 2 \\ &= 27 - 3 + 2 \\ &= 26. \end{aligned}$$

$$f(x) = 3x^2 - x + 2$$
$$f(-3) = 3(-3)^2 - (-3) + 2$$
$$= 27 + 3 + 2$$
$$= 32.$$

So the range of 'f' is $\{4, 26, 32\}$

Exercise 1-11.

1. If $f = \{(x, y) \mid y = 4x - 1\}$,
   find $f(0)$, $f(4)$ and $f(-2)$

2. If $g = \{(x, y) \mid y = -3x + 2\}$,
   find $g(1)$, $g(2)$ and $g(-3)$.

3. $f(x) = x^2 - 7$; find $f(2)$, $f(4)$ and $f(5)$.

4. D is the domain of the function given below. Find the range.

   (a) $f(x) = x^2 + 1$, $D = \{1, -1, 0\}$
   (b) $g(x) = x^2 - x + 1$ $D = \{2, 3, -5\}$
   (c) $r(x) = 2x^2 + 3$ $D = \{0, 1, 2\}$
   (d) $h(x) = 3x^2 + 2x + 7$ $D = \{1, -2, -3\}$
   (e) $k(x) = (2x - 3)^2$ $D = \{1, -1, 2, 5\}$

5. If $f(x) = 3x - 5$, and $g(x) = x^2 + 4$, $h(x) = 3x - 5$
   find the following.

   (a) $f(1) + g(2)$ (b) $f(-2) + g(3)$
   (c) $f(-4) + g(-5)$ (d) $f(-1) - g(-3)$
   (e) $g(-8) + h(-2)$ (f) $g(7) - h(-3)$
   (g) $g[h(-3)]$ (h) $h[g(4)]$

ANSWERS

1. -1, 15, -9, 4. (a) (2, 2, 1) (d) (12, 23, 40)
5. (a) 6 (d) -21 (g) 200.

REVIEW

1. A set is a collection of well-defined objects.

2. If every element of set A is an element of set B, then A is a sub-set of B. $[A \subseteq B]$

   ϕ is a subset of all sets.

   A set is a subset of itself.

   The set of all the subsets of a set is the power set of that set.

   If the caridinal number of a set is p, then the cardinal number of its power set is 2^p.

3. The set formed by all the elements common to any two sets is called the intersection of the two sets.

   The set formed by all the elements which belong to either or both of any two sets is called the union of those two sets.

   The set formed with all elements in the Universal set, other than the elements of a given set is called the complement of the given set.

4. (a) (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

   (ii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (d) $(A \cup B)' = A' \cap B'$

   (e) $(A \cap B)' = A' \cup B'$

5. If every element of a set is associated with one and only one element of another set, the relation is called a function. The first set is called the domain and the second set is called the co-domain.

   The 'range' is the subset of the co-domain and consists of the images only.

SELF-ASSESSMENT TEST

1. Find the set of rational numbers in the set of natural numbers.

2. $\{ x/2 < x < 7, x \in N \}$ —write the set in the Roster method.

3. $A = \{2, 4, 9, 12, 17\}$ $B = \{-5, -4, 0, 7\}$
   $C = \{-4, 4, 12, 18\}$

   Find (a) $A \cup B$ (b) $A \cup C$ (c) $B \cup C$ (d) $A \cap B$ (e) $A \cap C$ (f) $B \cap C$

4 Make use of the sets given above and verify the statements.

   (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

   (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

   (c) $n(B \cup C) = n(B) + n(C) - n(B \cap C)$

5. The universal set is the set of single digit natural numbers.
   $A = \{$ prime numbers $\}$

   What is A', the complement of A?

6. State De Morgan s laws.

   Verify those laws, with the help of some sets.

7. In a class of 45 pupils, 25 can sing, 20 can dance and 10 can sing as well as dance. How many can neither sing nor dance?

8. A salesman studies the drinking habits of people and sends to his office the following particulars.

   Number interviewed = 1,000

Those who Drinks	Number
Tea only	300
Coffee only	150
Cocoa only	200
Both tea and coffee	70

Both tea and cocoa	50
Both coffee and cocoa	100
All the three	20
None of the three	110

The salesman was dismissed from service by the Head Office. Find the reason.

9. Which of the following ordered pairs are functions ?

(a) { (4, 1), (4, 2), (4, 3), (4, 5) }

(b) { (1, 1), (1, 2), (2, 2), (1, 4) }

(c) { (2, 2), (3, 4), (4, 3), (1, 1) }

(d) { (1, 3), (2, 3), (3, 4), (3, 5) }

(e) { (4, 1), (3, 2), (4, 2) }

(f) { (6, 2), (5, 3), (4, 4) }

10. $f(x) = 2x^2 + x - 5$. Which of the ordered pairs is applicable to this function ? Find the range for the domain { 4, −3, −1 }.

NOT FOR EXAMINATION

MATHEMATICS CLUB—ACTIVITY I

Leonardo of Pisa, familiarly known as Fibonacci, presented a very interesting sequence to the world.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Note how the sequence has been formed.

Let us take any two consecutive numbers, e.g. 8 and 13.

Find their product. $8 \times 13 = 104$

Find the product of their neighbours $5 \times 21 = 105$

Find the product of two which are two places remote from them. $3 \times 34 = 102$

Product of numbers which are three places away from them. $2 \times 55 = 110$

Product of numbers which are four places away from them. $1 \times 89 = 89$

Take some groups of numbers like this and verify whether the following statements are true or false.

(a_n is the *n*th term in the series)

1. $a_n = a_{n-1} + a_{n-2}$
2. $a_n \times a_{n+1} = a_{n-1} \times a_{n+2} \pm 1$
3. $a_n \times a_{n+1} = a_{n-2} \times a_{n+3} \pm 2$
4. $a_n \times a_{n+1} = a_{n-3} \times a_{n+4} \pm 6$
5. $a_n \times a_{n+1} = a_{n-4} \times a_{n+5} \pm 15$
6. $a_{n-1} \times a_{n+1} = a_n^2 \pm 1$
7. $a_{n-2} \times a_{n+2} = a_n^2 \pm 1$
8. $a_{n-3} \times a_{n+3} = a_n^2 \pm 2^2$
9. $a_{n-4} \times a_{n+4} = a_n^2 \pm 3^2$
10. $a_{n-5} \times a_{n+5} = a_n^2 \pm 5^2$
11. $a_n^2 + a^2_{n+1}$ is also a Fibonacci number.
12. $a_1 + a_2 + a_3 + \dots + a_n = a_{n+2} - 1$

Take any two numbers, Add these two to form the third number. Add the 2nd number to the third and form the 4th. Add the 3rd and the 4th to form the 5th number and so on and form the series.

Then verify the facts mentioned above.

It has been found that the spiral pattern of the sun-flower, the leaves in a single stemmed plant and many other things in Nature have close connection with the Fibonacci sequence.

Try to know more about them.

2. NUMBER SYSTEM

REVISION

2.1. Real Numbers

Recall the following :

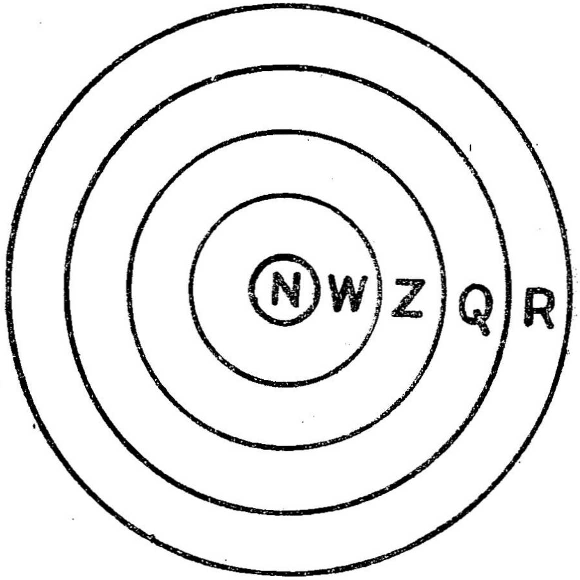
$N = \{1, 2, 3, \ldots\ldots\}$

$W = \{0, 1, 2, 3, \ldots\ldots\}$

$Z = \{\ldots\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\ldots\}$

$Q = \{\text{Rational numbers}\}$

$R = \{\text{Real numbers}\}$



$N \subset W \subset Z \subset Q \subset R$

Fig. 2-1

Exercise 2.1

1. State whether the following statements are true or false :

   (a) All natural numbers are whole numbers.

   (b) If a number is a whole number, then it is a natural number.

   (c) The product of two integers is a whole number.

   (d) Any integer can be written as a rational number.

   (e) All the numbers in the set of real numbers are irrational numbers.

   (f) The sum of a rational number and an irrational number is a real number.

2. Which of the following are rational and which are irrational ?

   (a) $\sqrt{7}$ (b) $\frac{-3}{4}$ (c) $\frac{\sqrt{3}}{4}$ (d) $\sqrt{16}$

   (e) $7\sqrt{3}$ (f) $\frac{-4\sqrt{25}}{3}$ (g) $\frac{4}{\sqrt{3}}$ (h) $3+4\sqrt{3}$

2.2. Properties of Real Numbers

If $a, b, c \in R$,

Addition Properties

1. $a + b \in R$ (Closure).
2. $a + b = b + a$ (Commutative).
3. $(a + b) + c = a + (b + c)$ (Associative).
4. $a + 0 = a$ (Identity)
5. $a + (-a) = 0$ (Inverse).

Multiplication Properties

6. $a \times b \in R$ (Closure).
7. $a \times b = b \times a$ (Commutative).
8. $(a \times b) \times c = a \times (b \times c)$ (Associative).
9. $a \times 1 = 1 \times a = a$ (Identity).
10. $a \times a^{-1} = a^{-1} \times a = 1$ (Inverse) [Note : $a \neq 0$].

Distributive Property

11. $a \times (b + c) = a \times b + a \times c.$

Exercise 2·2

Using the following numbers, verify the properties of the set of real numbers.

	a	b	c
1.	4	5	8
2.	-4	3	-6
3.	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$
4.	$-\frac{2}{3}$	$\frac{5}{8}$	$-\frac{3}{4}$
5.	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{10}$
6.	$2+\sqrt{3}$	$3-\sqrt{5}$	$3-\sqrt{3}$

2.3. The set of real numbers : Subsets

The following table shows the properties of the real numbers valid for its subsets.

Properties	N	W	Z	Q^+	Q	R
Addition						
1. Closure property	✓	✓	✓	✓	✓	✓
2. Associative property	✓	✓	✓	✓	✓	✓
3. Additive identity (0)	✗	✓	✓	✓	✓	✓
4. Additive inverse	✗	✗	✓	✓	✓	✓
5. Commutative property	✓	✓	✓	✓	✓	✓

Properties	N	W	Z	Q^+	Q	R
Multiplication						
1. Closure property	✓	✓	✓	✓	✓	✓
2. Associative property	✓	✓	✓	✓	✓	✓
3. Multiplicative identity(1)	✓	✓	✓	✓	✓	✓
4. Multiplicative inverse	✗	✗	✗	✓	✓	✓
5. Commutative property	✓	✓	✓	✓	✓	✓
6. Distributive property	✓	✓	✓	✓	✓	✓

Exercise 2.3

1. Which of the following are true and which are false ?

   (a) The set of natural numbers has additive identity.

   (b) The set of whole numbers has additive inverse.

   (c) The set of integers has closure property under multiplication.

   (d) There exists the multiplicative identity in the set of natural numbers.

   (e) There exists a multiplicative inverse for every element of the set of rational numbers.

2. Is x a member of N if $a + x = a$?

3. Does x belong to Z if $a \times x = 1$?

4. If $a, b \in R$, $a + b = b + a$. Is this true ?

5. Is $a(b + c) = ab + ac$ true if $a, b, c \in Z$?

2·4. Arithmetic progression

(1) 4, 7, 10, 13, ...

(2) 5, 10, 15, 20, ...

(3) 8, 6, 4, 2, 0, ...

(4) 5, 11, 17, 23, ...

Study the sequences given above. The difference between any two consecutive terms is the same in each of the sequences.

The difference in sequence (1) is 3

The difference in sequence (2) is 5

The difference in sequence (3) is -2

The difference in sequence (4) is 6

Such sequences where the consecutive terms have the same difference are called arithmetic sequences, more familiarly known as Arithmetic progressions, *A.P.* for short

In general, $a,\ a + d,\ a + 2d,\ \ldots$ is an *A P.*

In this, the first term is a. The difference between any two consecutive terms is d.

The first term	$= a$
2nd term	$= a + d = a + (2 - 1)\,d$
3rd term	$= a + 2d = a + (3 - 1)\,d$
4th term	$= a + 3d = a + (4 - 1)\,d$
... ...	... ...
... ...	... ...
... ...	... ...
nth term	$= a + (n - 1)\,d$

$$t_n = a + (n - 1)\,d$$

Example 1.

Find out the 32*nd* term of the *A.P.* 3, 7, 11, ...

Solution :

$$t_n = a + (n - 1)\,d$$

given $\quad a = 3; \quad d = 4; \quad n = 32$

$$t_{32} = 3 + (32 - 1)\,(4)$$
$$= 3 + (31)\,(4)$$
$$= 3 + 124$$
$$= 127$$

32 *nd* term is 127.

Let l be the last term of the *A. P.* $a, a + d, a + 2d, \ldots$ and let it be the n th term

i.e. $t_n = l$

$$a + (n - 1)\,d = l$$
$$(n - 1)\,d = l - a$$
$$n - 1 = \frac{l - a}{d}$$

$$\boxed{n = \frac{l - a}{d} + 1}$$

Example 2

98 is the nth term of the A.P. 3, 8, 13, ... Find n.

Solution :

$$n = \frac{l - a}{d} + 1$$

given $l = 98, \; a = 3, \; d = 5$

$$n = \frac{98 - 3}{5} + 1 = \frac{95}{5} + 1$$
$$= 19 + 1$$
$$= 20.$$

98 is the 20th term.

Exercise 2.4

1. Write the next three terms in the following A.P.

   (a) 5, 12, 19, ...

   (b) 7, 4, 1, ...

   (c) 3, 3·5, 4, 4·5, ...

2. In the following, the first term and the common difference of an A.P. are given. Write the first four terms of the A.P.

	1st term	Common difference
(a)	5	4
(b)	3	2
(c)	−7	3
(d)	73	−2
(e)	10	−5

3. The first term of an A.P. is a and the common difference is d. Find the 12th, 35th and 100th term, given the values of a and d.

   (a) $a = 6,$ $d = 5$

   (b) $a = 4,$ $d = 7$

   (c) $a = 8,$ $d = 1\ 5$

4. How many terms are there in the A.P.

   8, 11, 14,..., 95 ?

5. Find the number of terms in the A P.

   11, 6, 1...., −54.

6. How many terms are there in the A.P.

   2, 3·5, 5, ..., 62 ?

7. Find the value of x if $3x + 1$, $5x - 1$, $5x + 1$ form an A.P.

8. The 12th term of an A.P. is 61. The common difference is 5. Find the A.P.

9. The 10th term of an A.P. is 31. The 15th term is 46. Find the A.P.

ANSWERS

1. (a) 26, 33, 40 (b) −2, −5, −8 (c) 5, 5·5, 6.

2. (a) 5, 9, 13, 17 (b) 3, 5, 7, 9 (c) −7, −4, −1, 2 (d) 73, 71, 69, 67 (e) 10, 0, −5.

3. (a) 61, 176, 501 (b) 81, 242, 697 (c) 24·5, 59, 156·5.

4. 30. 5. 14. 6. 41. 7. $x = 2$. 8. 6, 11, 16, ...

9. 4, 7, 10, ...

2·5. Arithmetic Progression : Sum to n terms

Recall : Sum to n terms of an A.P., where the first term is a and common difference is d, is

$$S_n = \frac{n}{2} \{2a + (n - 1) d\}$$

$$\boxed{S_n = \frac{n}{2} \{a + l\}}$$ (where l is the last term)

Example 1 :

Find the sum of the A P.

$5 + 8 + 11 + \ldots$ upto 30th term.

Solution :

$$S_n = \frac{n}{2} \{ 2a + (n - 1) d \}$$

$$n = 30, a = 5, d = 3.$$

$$S_{30} = \frac{30}{2} \{ (2 \times 5) + (30 - 1)\, 3 \}$$
$$= 15 \{ 10 + 29 \times 3 \}$$
$$= 15 \{ 10 + 87 \}$$
$$= 15 \times 97$$
$$= 1455.$$

Example 2 :

Find the sum of the A.P.

$$4 + 9 + 14 + \ldots + 199.$$

Solution :

First we must find the value of n.

$$n = \frac{l - a}{d} + 1$$

Substitute $l = 199$, $a = 4$, $d = 5$

$$n = \frac{199 - 4}{5} + 1$$
$$= \frac{195}{5} + 1$$
$$= 39 + 1$$
$$= 40.$$

$$S_n = \frac{n}{2} (a + l)$$
$$S_{40} = \frac{40}{2} (4 + 199)$$
$$= 20 \times 203$$
$$= 4060.$$

Exercise 2·5

Find the sum :

1. $3 + 7 + 11 + \ldots$ up to 25 terms.
2. $9 + 12 + 15 + \ldots$ up to 30 terms.
3. $7 + 5 + 3 + \ldots$ up to 15 terms.

4. $2 + 4{\cdot}5 + 7 + \dots$ up to 40 terms.

5. $5 + 6{\cdot}5 + 8 + \dots$ up to 60 terms

6. $7 + 12 + \dots + 152$

7. $6 + 10 + 14 + \dots + 202$

8. $5 + 3 + \dots + (-53)$

9. $3 + 4{\cdot}5 + \dots + 18$

10. Find the sum of all whole numbers divisible by 7 between 200 and 400.

ANSWERS

1.	1275	2.	1575	3.	−105	4.	2030
5.	2955	6.	2385	7.	5200	8.	−720
9.	115·5	10.	8729.				

2.6. Arithmetic progressions : Sum of odd natural numbers

Given : 1, 3, 5, .. ; $a = 1$; $d = 2$

Sum of the first n odd natural numbers

$$S_n = \frac{n}{2}\,[2{\cdot}1 + (n-1)\,2] = n^2$$

Example 1

The sum of the first 12 odd natural numbers is 12^2

$= 144$

Example 2

Find the sum :

$1 + 3 + 5 + \dots + 51$

Solution :

$$1 + 3 + 5 + \dots + 51 = \left(\frac{51 + 1}{2}\right)^2$$

$$= \left(\frac{52}{2}\right)^2$$

$$= 26^2$$

$$= 676.$$

Example 3 :

Find the sum :

$$13 + 15 + \dots + 47.$$

Solution :

$$1 + 3 + 5 + \dots + 47 = \left(\frac{47 + 1}{2}\right)^2$$

$$= \left(\frac{48}{2}\right)^2$$

$$= 24^2$$

$$= 576.$$

$$1 + 3 + 5 + \dots + 11 = \left(\frac{11 + 1}{2}\right)^2$$

$$= \left(\frac{12}{2}\right)^2$$

$$= 6^2$$

$$= 36.$$

$$13 + 15 + \dots + 47 = (1 + 3 + \dots + 47) - (1 + 3 + \dots + 11)$$

$$= 576 - 36$$

$$= 540.$$

Exercise 2·6

Find the sum of:

1. The first fifteen odd natural numbers
2. The first twenty odd natural numbers
3. The first twentyfive odd natural numbers
4. The first 40 odd natural numbers
5. The first 32 odd natural numbers
6. $1 + 3 + 5 + \dots + 73$

7. $1 + 3 + 5 + \dots + 101$
8. $1 + 3 + 5 + \dots + 143$
9. $23 + 25 + \dots + 111$
10. $47 + 49 + 51 + \dots + 307$

ANSWERS

1.	225	2.	400	3.	625	4.	1600	5.	1024
6.	1369	7.	2601	8.	5184	9.	3015	19.	23187.

2.7. Natural Numbers : Sum of squares

Observe the following :

$$1^2 = 1 = \frac{1\times 2\ (2\times 1+1)}{6}$$

$$1^2 + 2^2 = 5 = \frac{2\times 3\ (2\times 2+1)}{6}$$

$$1^2 + 2^2 + 3^2 = 14 = \frac{3\times 4\ (2\times 3+1)}{6}$$

$$1^2 + 2^2 + 3^2 + 4^2 = 30 = \frac{4\times 5\ (2\times 4+1)}{6}$$

Similarly we can conclude

$$1^2 + 2^2 + \dots + n^2 = \frac{n\ (n+1)\ (2n+1)}{6}$$

> Sum of squares of the first n natural numbers
>
> $$\sum n^2 = \frac{n\ (n+1)\ (2n+1)}{6}$$

Example 1 :

Find the sum, using formula

$$1^2 + 2^2 + 3^2 + \dots + 12^2$$

Solution :

$$\sum n^2 = \frac{n\ (n+1)\ (2n+1)}{6}$$

Substitute $n = 12$

$$1^2 + 2^2 + \ldots + 12^2 = \frac{12\,(12+1)\,(2\times 12+1)}{6}$$

$$= \frac{12 \times 13 \times 25}{6}$$

$$= 650$$

Example 2 :

Find the sum :

$11^2 + 12^2 + \ldots + 30^2$

$$1^2 + 2^2 + \ldots + 30^2 = \frac{30\,(30+1)\,(2\times 30+1)}{6}$$

$$= \frac{30 \times 31 \times 61}{6}$$

$$= 9455.$$

$$1^2 + 2^2 + \ldots + 10^2 = \frac{10\,(10+1)\,(2\times 10+1)}{6}$$

$$= \frac{10 \times 11 \times 21}{6}$$

$$= 385.$$

$$11^2+12^2+\ldots+30^2 = (1^2+2^2+\ldots+30^2) - (1^2+2^2+\ldots+10^2)$$

$$= 9455 - 385$$

$$= 9070.$$

Exercise 2·7

Find the sum, using formula.

1. $1^2 + 2^2 + \ldots + 20^2$
2. $1^2 + 2^2 + \ldots + 25^2$
3. $1^2 + 2^2 + \ldots + 40^2$
4. $1^2 + 2^2 + \ldots + 50^2$
5. $1^2 + 2^2 + \ldots + 84^2$
6. $9^2 + 10^2 + \ldots + 17^2$

7. $15^2 + 16^2 + ... + 27^2$
8. $17^2 + 18^2 + ... + 30^2$
9. $21^2 + 22^2 + ... + 35^2$
10. $28^2 + 29^2 + ... + 60^2$

ANSWERS

1.	2870	2.	5525	3.	22140	4.	42925	5.	201110
6.	1581	7.	5915	8.	7959	9.	12040	10.	66880

2.8. Natural Numbers : Sum of Cubes

$$1^3 = 1 = \left[\frac{1 \times (1+1)}{2}\right]^2$$

$$1^3 + 2^3 = 1 + 8 = 9 = \left[\frac{2 \times (2+1)}{2}\right]^2$$

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 = \left[\frac{3 \times (3+1)}{2}\right]^2$$

Similarly

$$\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Note : $\Sigma n^3 = [\Sigma n]^2$

Example 1 :

Find the sum, using formula.

$1^3 + 2^3 + 3^3 + .. 12^3$

Solution :

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Substitute $n = 12$

$$1^3 + 2^3 + 3^3 + ... + 12^3 = \left[\frac{12 \times 13}{2}\right]^2$$

$$= (6 \times 13)^2$$

$$= 78^2$$

$$= 6084$$

Example 2 :

Find the sum of $11^3 + 12^3 + \dots + 20^3$

Solution :

$$1^3 + 2^3 + \dots + 20^3 = \left[\frac{20 \times 21}{2}\right]^2$$
$$= (10 \times 21)^2$$
$$= 210^2$$
$$= 44100$$

$$1^3 + 2^3 + \dots + 10^3 = \left[\frac{10 \times 11}{2}\right]^2$$
$$= (5 \times 11)^2$$
$$= 55^2$$
$$= 3025.$$

$$\therefore 11^3 + 12^3 + \dots + 20^3 = (1^3 + 2^3 + \dots + 20^3) - (1^3 + 2^3 + \dots + 10^3)$$
$$= 44100 - 3025$$
$$= 41075.$$

Exercise 2.8

Find the sum of the following, using formula :

1. $1^3 + 2^3 + \dots + 15^3$
2. $1^3 + 2^3 + \dots + 20^3$
3. $1^3 + 2^3 + \dots + 32^3$
4. $1^3 + 2^3 + \dots + 47^3$
5. $1^3 + 2^3 + \dots + 51^3$
6. $7^3 + 8^3 + \dots + 16^3$
7. $6^3 + 7^3 + \dots + 25^3$
8. $12^3 + 13^3 + \dots + 45^3$
9. $15^3 + 16^3 + \dots + 70^3$
10. If $1 + 2 + \dots + 24 = 300$, find the sum of $1^3 + 2^3 + \dots + 24^3$

ANSWERS

1. 14400	2. 44100	3. 278784	4. 1272384
5. 1758276	6. 18055	7. 105400	8. 1066869
9. 6164200	10. 90000.		

2·9. Geometric Progression

Look at the series 4, 12, 36, 108, ... Note its formation. Every succeeding number is three times the preceding number. Such a sequence is called a geometric progression, G. P. for short.

Example

1. 5, 10, 20, 40, ...
2. 2, -4, 8, -16, ...
3. 5, $\frac{5}{3}$, $\frac{5}{9}$, $\frac{5}{27}$, ...

The ratio between a number and its predecessor is called **the common ratio** of the G. P.

In example (1), the common ratio is

$$\frac{10}{5} = \frac{20}{10} = \frac{40}{20} = 2$$

In example (2), the common ratio is

$$\frac{-4}{2} = \frac{8}{-4} = \frac{-16}{8} = -2$$

In example (3), the common ratio is

$$\frac{\frac{5}{3}}{5} = \frac{\frac{5}{9}}{\frac{5}{3}} = \frac{\frac{5}{27}}{\frac{5}{9}} = \frac{1}{3}$$

Example 1

Find the next three terms in the G. P. 3, 6, 12, ...

Solution :

The common ratio is $\frac{6}{3} = 2$

$\therefore$ The next three terms are 24, 48, 96.

To find the n th term in a G. P.

a is the first term in the G. P.

$a, ar, ar^2, ar^3, \ldots$

First term $= a \quad = ar^{1-1}$

2nd term $= ar \quad = ar^{2-1}$

3rd term $= ar^2 = ar^{3-1}$

... ...

... ...

n th term $= ar^{n-1}$

$$t_n = ar^{n-1}$$

Example 2

Find the 10th term in the G. P.

$\frac{1}{2}, 1, 2, 4, \ldots\ldots\ldots$

Solution :

$a = \frac{1}{2}, r = 2; n = 10$

Substituting these in the formula $t_n = ar^{n-1}$, we get

$$t_{10} = \left(\frac{1}{2}\right)(2)^{10-1}$$

$$= \frac{1}{2} \times 2^9$$

$$= 2^{-1} \times 2^9$$

$$= 2^8$$

$$= 256.$$

Exercise 2·9

1. Find the common ratio of the following G. P.

(a) 6, 18 54, ...,

(b) 7, -14, 28, ...

(c) $\frac{1}{5}$, 1, 5,...

(d) 32, 16, 8, ...

(e) 16, -4, 1, ...

(f) 1, 0·2, 0·04, ...

2. Find the next 3 terms in the following G. P.

(a) 5, 10, 20, ... (c) -4, 12. -36, ...

(b) 81, 27, 9, ... (d) $\frac{1}{7}$, 1, 7, ...

3. In a G. P. the first term is a. The common ratio is r. Find the first 4 terms in the progression.

(a) $a = 15$ $r = 2$ (d) $a = 125$ $r = \frac{1}{5}$

(b) $a = -\frac{1}{4}$ $r = 4$ (e) $a = -3$ $r = -3$

(c) $a = 0{\cdot}2$ $r = 3$ (f) $a = 4$ $r = -\frac{1}{2}$

4. In a G. P., the first term is a. The common ratio is r. n th term is t_n.

Find the following :

(a) $a = 6$ $r = 2$ $t_5 = ?$

(b) $a = -3$ $r = \frac{1}{3}$ $t_7 = ?$

(c) $a = -1$ $r = 4$ $t_6 = ?$

(d) $a = 25$ $r = \frac{2}{5}$ $t_4 = ?$

5. Find the value of x in the G. P.

3, x, 27.

6. If $\frac{1}{3}$, k, $\frac{1}{12}$ form a G.P., find the value of k.

ANSWERS

1. (a) 3 (b) --2 (c) 5 (d) $\frac{1}{2}$ (e) $-\frac{1}{4}$ (f) 0.2

2. (a) 40, 80, 160 (b) 3, 1, $\frac{1}{3}$

(c) 108, -324, 972. (d) 49, 343, 2401.

3. (a) 15, 30, 60, 120. (d) 125, 25, 5, 1

(b) $-\frac{1}{4}, -1, -4, -16$ (e) $-3, 9, -27, 81$

(c) 0·2, 0 6, 1·8, 5·4 (f) $4, -2, 1, -\frac{1}{2}$

4. (a) 96 (b) $-\frac{1}{243}$ (c) -1024 (d) $\frac{8}{5}$

5. ± 9 6. $\pm \frac{1}{6}$

2·10. Geometric Progression : Means

Example :

The 7th term of a G.P. is 128 and the 10th term is 1024. Find the 8th and 9th terms.

Solution :

$$t_n = ar^{n-1}$$

$$t_7 = ar^6 = 128 \qquad \ldots \qquad (1)$$

$$t_{10} = ar^9 = 1024 \qquad \ldots \qquad (2)$$

Divide equation (2) by equation (1)

$$\frac{ar^9}{ar^6} = \frac{1024}{128} = 8$$

$$r^3 = 8$$

$$r = \sqrt[3]{8}$$

$$r = 2$$

$$t_8 = 128 \times 2 = 256$$

$$t_9 = 256 \times 2 = 512$$

t_8, t_9 are the geometric between t_7 and t_{10}

Exercise 2-10.

1. In a G.P., the 5th term is 48 and the 8th term is 384. Find the 6th and 7th terms.

2. In a G.P., the 4th term is 108 and the 6th term is 972. Find the 5th term.

3. The 3rd term of a G.P. is 80. The 6th term is 5120. Find the 4th and the 5th terms.

ANSWERS

1. 96, 192 2. ± 324 3. 320, 1280.

REVIEW

1. Table in 2·3 shows the properties of number sets regarding addition and multiplication operations.

2. The set of numbers which are formed according to certain definite rules form a sequence.

3. A sequence which has a constant difference between two consecutive terms is called an Arithmetic Progression (A.P.)

   In the A.P. $a, a + d, a + 2d, \ldots$ the first term is a, common difference is d, n th term $t_n = a + (n-1)\,d$.

   The sum of the first n terms $S_n = \frac{n}{2}\,[2a + (n - 1)\,d]$

   If the first term is a, last term is l and the common difference is d, then the total number of terms in the sequence, $n = \frac{l - a}{d} + 1$

   The sum of all the terms is $\frac{n}{2}\,(a + l)$

4. (a) $\Sigma\, n = 1 + 2 + 3 + \ldots + n = \frac{n\,(n + 1)}{2}$

   (b) $\Sigma\,(2n - 1) = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

   (c) $1 + 3 + 5 + \ldots + l = \left(\frac{l + 1}{2}\right)^2$

   (d) $\Sigma\, n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$
   $= \frac{n\,(n + 1)\;(2n + 1)}{6}$

   (e) $\Sigma\, n^3 = 1^3 + 2^3 + 3^3 + \quad + n^3$
   $= \left(\frac{n\,(n + 1)}{2}\right)^2 = (\Sigma\, n)^2$

5. If the ratio of the two consecutive terms of a sequence is constant, then the sequence is said to be a Geometric Progression. In the series $a, ar, ar^2, \ldots$ the first term is a, common ratio is r and the nth term $t_n = ar^{n-1}$.

SELF ASSESSMENT TEST

1. (a) What is the difference between the properties of the rational numbers and those of the positive rational numbers ?

   (b) What are the number sets which have no additive inverse ?

   (c) What are the number sets which have no multiplicative inverse ?

   (d) Which is the set that has multiplicative inverse but has no additive inverse ?

   (e) Which is the set that has no additive identity ?

2. The first term of an A.P. is 8 and the common difference is 5. Write the first five terms of the A.P. Find the 20th term.

3. The 10th term of an A.P. is 35 and the 25th term is 80. What is the first term ? What is the common difference?

4. In a Geometric progression, the first term is 8 and the common ratio is -2. Find the 6th and the 11th terms.

5. In a G P. the 5th term is 3. The 8th term is $\frac{1}{9}$. What is the first term ? What is the common ratio ? Find the terms between the 5th term and the 8th.

6. Find the sum of the series.

   (a) $1 + 2 + 3 + \ldots + 24$

   (b) $1 + 3 + 5 + 7 + \ldots + 91$

   (c) $21 + 23 + 25 + \ldots + 101$

   (d) $1^2 + 2^2 + 3^2 + \quad + 25^2$

   (e) $11^2 + 12^2 + 13^2 + \ldots + 40^2$

   (f) $1^3 + 2^3 + 3^3 + \ldots + 20^3$

(g) $8^3 + 9^3 + 10^3 + \ldots + 20^3$

(h) The first 40 terms of the A.P. 25, 22, 19, ..

(i) The first term of the A.P. is 25. The last term is 425. The common difference is 4.

ANSWERS

1. (a) In the set of positive rational numbers there is no additive inverse

   (b) The set of natural numbers, whole numbers and positive rational numbers.

   (c) The set of natural numbers, whole numbers and integrs.

   (d) Set of positive rational numbers.

   (e) Set of natural numbers.

2. 103 3. 8, 3, 4. -256, 8192. 5. 243, $\frac{1}{3}$, 1, $\frac{1}{3}$.

6. 300, 2116. 2501, 5525, 21755, 44100, 43316, -1340, 22725.

NOT FOR EXAMINATION

MATHEMATICS CLUB—ACTIVITY 2

Work out and take:

A son wrote a letter to his father asking for money. The father signed a cheque and wrote the following:

```
  S A V E
+ M O R E
---------
M O N E Y
```

He asked his son to enter the amount in the cheque, after substituting numbers for the letters found in the instruction. Every letter represents a number. No two different letters represent the same number. Find the maximum amount that the son can draw.

3. ALGEBRA

3-1. Identities : Revision

Let us recall some of the identities which we learnt in the Ninth Standard.

1. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

2. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

3. $(a-b)^3 = a^3 - 3a^2b^2 + 3ab^2 - b^3$

4. $(x+a)(x+b)(x+c)$
 $= x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc$

5. $(a+b)(a^2 - ab + b^2) = a^3 + b^3$

6. $(a-b)(a^2 + ab + b^2) = a^3 - b^3$

Exercise 3-1

I. Give the expansions of: (Mental sums)

1. $(x + y + z)^2$
2. $(p + q - r)^2$
3. $(k - l - m)^2$
4. $(2a - 5b + 4c)^2$
5. $(x + y)^3$
6. $(4x + 3)^3$
7. $(p - q)^3$
8. $(x - 4)^3$
9. $(5x - 3y)^3$
10. $\left(\frac{x}{y} + \frac{y}{x}\right)^3$
11. $(x+3)(x+4)(x+5)$
12. $(p-1)(p-2)(p-3)$
13. $(m+7)(m+1)(m+3)$
14. $(a-4)(a+3)(a-5)$

15. $(x+y)(x^2-xy+y^2)$ 16. $(a+3)(a^2-3a+9)$

17. $(2x+3y)(4x^2-6xy+9y^2)$ 18. $(p^2-q)(p^2+pq+q)$

19. $(k+4)(k^2-4k+16)$

20. $(3x-5y)(9x^2+15xy+25y^2)$

II. If $x+\frac{1}{x}=3$, find the values of $x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$

III. If $a-\frac{1}{a}=4$, find the values of $a^2+\frac{1}{a^2}$ and $a^3-\frac{1}{a^3}$

ANSWERS

II 7, 18 III 18, 76

3-2. Factorisation : (a^2-b^2) type

Recall : $a^2-b^2=(a+b)(a-b)$

Example

Find the factors of $25p^2-49q^2$

Solution :

$$25p^2-49q^2=(5p)^2-(7q)^2$$
$$=(5p+7q)(5p-7q)$$

Exercise 3-2

(Mental sums)

Find the factors of:

(1) x^2-y^2 (2) m^2-n^2 (3) p^2-q^2

(4) c^2-d^2 (5) x^2-1 (6) p^2-9

(7) m^2-25 (8) k^2-49 (9) $16-c^2$

(10) $36-x^2$ (11) $64-p^2$ (12) $25x^2-16$

(13) $49a^2-100$ (14) $81m^2-4$ (15) $4x^2-25y^2$

(16) $81a^2-16b^2$ (17) $121a^2-100b^2$ (18) $9k^2-49l^2$

3-3. Factorisation : $x^2 + px + q$ type (Revision)

Recall : $x^2 + (a + b)x + ab = (x + a)(x + b)$

Example :

Find the factors of $x^2 + 9x + 14$ | $14 = 2 \times 7$

Solution :

$x^2 + 9x + 14 = (x + 7)(x + 2)$ | $9 = 2 + 7$

Exercise 3-3.

(Mental sums)

Find the factors of :

(1) $x^2 + 3x + 2$ (2) $a^2 + 4a + 3$

(3) $p^2 + 11p + 24$ (4) $m^2 + 5m + 6$

(5) $a^2 + 7a + 6$ (6) $x^2 + 8x + 12$

(7) $x^2 + 10x + 9$ (8) $c^2 + 6c + 8$

(9) $p^2 + 21p + 110$ (10) $x^2 + 13x + 42$

(11) $x^2 - 7x + 10$ (12) $a^2 - 5a + 4$

(13) $p^2 - 10p + 21$ (14) $x^2 - 11x + 18$

(15) $k^2 - 12k + 35$ (16) $x^2 - 10x + 24$

(17) $m^2 - 11m + 28$ (18) $k^2 - 9k + 14$

(19) $x^2 - 20x + 96$ (20) $y^2 - 15y + 50$

3-4. Factorisation : $(x^2 + px - q)$ type : (Revision)

Recall : $x^2 + (a-b)x - ab = (x + a)(x - b)$

Example :

$x^2 + 2x - 35 = (x + 7)(x - 5)$ | $35 = 7 \times 5$, $2 = 7 - 5$

Exercise 3-4

Find the factors of :

(1) $x^2 + x - 6$ (2) $p^2 + p - 12$

(3) $m^2 + m - 2$ (4) $y^2 + 2y - 8$

(5) $x^2 + 2x - 15$	(6) $k^2 + 4k - 32$
(7) $x^2 + 3x - 10$	(8) $y^2 + 3y - 18$
(9) $c^2 + c - 20$	(10) $x^2 + 2x - 120$
(11) $x^2 - 3x - 4$	(12) $a^2 - a - 56$
(13) $m^2 - 5m - 14$	(14) $c^2 - 4c - 45$
(15) $p^2 - 4p - 77$	(16) $k^2 - 11k - 26$
(17) $x^2 - 11x - 60$	(18) $a^2 - 3a - 28$
(19) $p^2 - 5p - 66$	(20) $x^2 - 2x - 143$

3-5. Factorisation : $(ax^2 + bx + c)$ type

Multiply $(2x + 5)(3x + 4)$

$$\begin{array}{r} 2x + 5 \\ 3x + 4 \\ \hline 8x + 20 \\ 6x^2 + 15x \quad \\ \hline 6x^2 + 23x + 20 \end{array}$$

Note : $6 \times 20 = 120$

$8 \times 15 = 120.$

To factorise $ax^2 + bx + c$,

1. Multiply the coefficient of x^2 and the constant term (ac)
2. Find pairs of factors of ac
3. Select the pair which, when added, is equal to the coefficient of x i.e. b.

Example 1 :

Find the factors of : $2x^2 + 11x + 12$.

Solution :

$2x^2 + 11x + 12$

$= 2x^2 + 8x + 3x + 12$

$= 2x(x + 4) + 3(x + 4)$

$= (x + 4)(2x + 3)$

Note :

$12 \times 2 = 24$

$8 \times 3 = 24$

$8 + 3 = 11$

$\therefore 11x = 8x + 3x$

Exercise 3-5

Find the factors of :

(1) $2x^2 + 3x + 1$

(2) $2a^2 + 5a + 2$

(3) $2x^2 + 7x + 5$

(4) $3a^2 + 10a + 7$

(5) $2m^2 + 11m + 5$

(6) $12x^2 + 7x + 1$

(7) $2x^2 + 7x + 3$

(8) $6p^2 + 5p + 1$

(9) $2x^2 + 9x + 10$

(10) $3a^2 + 10a + 8$

(11) $7x^2 + 25x + 12$

(12) $15d^2 + 11d + 2$

(13) $6m^2 + 29m + 9$

(14) $12x^2 + 31x + 20$

(15) $2x^2 + 9x + 4$

(16) $4x^2 + 13x + 10$

(17) $4x^2 + 13x + 3$

(18) $6x^2 + 23x + 10$

(19) $10x^2 + 23x + 9$

(20) $8x^2 + 27x + 9$

ANSWERS

(1) $(x + 1)(2x + 1)$

(2) $(2a + 1)(a + 2)$

(7) $(2x + 1)(x + 3)$

(13) $(3m + 1)(2m + 9)$

(16) $(x + 2)(4x + 5)$

(20) $(x + 3)(8x + 3)$

3-6. Factorisation : $(ax^2 - bx + c)$ type

Example :

Find the factors of: $4x^2 - 16x + 15$

Solution :

$4x^2 - 16x + 15$

$= 4x^2 - 10x - 6x + 15$

$= 2x(2x \quad 5) - 3(2x - 5)$

$= (2x - 5)(2x - 3)$

Note :

$15 \times 4 = 60$

$60 = 10 \times 6$

$(-10) + (-6) = -16$

$-16x = -10x - 6x$

Exercise 3-6

Find the factors of :

(1) $8x^2 - 10x + 3$ (2) $6p^2 - 17p + 5$

(3) $10m^2 - 19m + 7$ (4) $3c^2 - 11c + 6$

(5) $2d^2 - 15d + 25$ (6) $2x^2 - 25x + 50$

(7) $3y^2 - 31y + 56$ (8) $15c^2 - 19c + 6$

(9) $4x^2 - 23x + 15$ (10) $3a^2 - 10a + 8$

ANSWERS

(1) $(2x-1)(4x-3)$ (3) $(5m-7)(2m-1)$

(8) $(5c-3)(3c-2)$ (10) $(a-2)(3a-4)$

3-7 Factorisation : $(ax^2 + bx - c)$ type

Example :

Find the factors of : $6x^2 + 7x - 20$

Solution :

$6x^2 + 7x - 20$

$= 6x^2 + 15x - 8x - 20$

$= 3x(2x + 5) - 4(2x + 5)$

$= (2x + 5)(3x - 4)$

Note :

$20 \times 6 = 120$

$120 = 15 \times 8$

$(+15) + (-8) = +7$

$+7x = +15x - 8x$

Exercise 3-7

Find the factors of :

(1) $2a^2 + 7a - 4$ (2) $2a^2 + a - 3$

(3) $3x^2 + 7x - 20$ (4) $2p^2 + 5p - 12$

(5) $3x^2 + 10x - 48$ (6) $5a^2 + 42a - 27$

(7) $6x^2 + x - 15$ (8) $18a^2 + 9a - 35$

(9) $20m^2 + m - 12$ (10) $15a^2 + 17a - 4$

ANSWERS

(1) $(a+4)$ $(2a-1)$ (2) $(2a+3)$ $(a-1)$

(4) $(p+4)$ $(2p-3)$ (9) $(5m+4)$ $(4m-3)$

3.8 Factorisation : $(ax^2 - bx - c)$ type

Example 1

Find the factors of : $3x^2 - 2x - 8$

Solution :

$3x^2 - 2x - 8$

$= 3x^2 - 6x + 4x - 8$

$= 3x(x-2) + 4(x-2)$

$= (x-2)(3x+4)$

Note :

$(+8) \times (+3) = +24$

$24 = 6 \times 4$

$(-6) + (+4) = -2$

$-2x = 6x + 4x$

Exercise 3-8

Find the factors of :

(1) $9x^2 - 5x - 4$

(2) $5x^2 - x - 18$

(3) $8x^2 - 6x - 5$

(4) $6x^2 - 5x - 6$

(5) $7x^2 - x - 6$

(6) $7p^2 - 10p - 8$

(7) $9d^2 - 9d - 4$

(8) $12a^2 - 7a - 10$

(9) $8l^2 - 6l - 9$

(10) $28 - 31a - 5a^2$

ANSWERS

(1) $(9x+4)$ $(x-1)$ (5) $(x-1)$ $(7x+6)$

(8) $(4a-5)$ $(3a+2)$ (10) $(4-5a)$ $(7+a)$

3-9 Remainder theorem

We know how to divide a polynomial by another polynomial, finding the quotient and the remainder.

Let us divide the polynomial $x^3 - 2x^2 + 3x - 4$ by $(x-1)$ $(x-2)$ and $(x+2)$ and find the remainders.

They are -2, 2 and -14 respectively.

If $P(x) = x^3 - 2x^2 + 3x - 4$, let us find $P(1)$, $P(2)$, $P(-2)$ and tabulate our results.

Divisor	Remainder	Value of $P(x)$
$x-1$	-2	$P(1) = 1-2+3-4 = -2$
$x-2$	2	$P(2) = 8-8+6-4 = 2$
$x+2$	-26	$P(-2) = -8-8-6-4 = -26$

From these, we conclude that when we divide $P(x)$ by $(x-1)$, the remainder is $P(1)$.

When we divide $P(x)$ by $(x-2)$, the remainder is $P(2)$.

When we divide $P(x)$ by $(x+2)$, the remainder is $P(-2)$.

So we can say that when we divide $P(x)$ by $(x-a)$, the remainder is $P(a)$.

The Remainder Theorem :

If a polynomial $P(x)$ over a set of real numbers is divided by the binomial expression $(x - a)$, $a \in R$, then the remainder is $P(a)$.

Exercise 3-9

Find the remainder when the following polynomials are divided by the binomials given against them.

Polynomials	Divisors
(1) x^2-7x+3	$(x+1)$, $(x-1)$, $(x+2)$
(2) x^3+3x^2+2	$(x-1)$, $(x-2)$, $(x+3)$
(3) x^3-4x+5	$(x+1)$, $(x+2)$, $(x+3)$
(4) x^3+x^2+x+1	$(x-1)$, $(x+1)$, $(x-2)$
(5) $2x^3-4x^2+5x-3$	$(x+2)$, $(x-1)$, $(x-3)$

ANSWERS

(1) 11, −3, 21 **(4) 4, 0, 15**

3-10. Remainder Theorem : Miscellaneous problems.

Example 1 :

The remainder is 13 when the polynomial $2x^2 - ax + 11$ is divided by $(x-2)$. Find the value of a.

Solution :

$P(x) = 2x^2 - ax + 11$

Remainder when divided by $(x - 2)$,

$$P(2) = 2(2)^2 - a(2) + 11 = 13$$
$$8 - 2a + 11 = 13$$
$$-2a = 13 - 8 - 11$$
$$-2a = -6$$
$$a = 3.$$

Exercise 3-10

Some polynomials, the divisors and the remainders are given below. Find the values of a.

	Polynomial	Divisor	Remainder
(1)	$4x^2 - 5x + a$	$x - 1$	6
(2)	$2x^2 - 4x + a$	$x + 1$	9
(3)	$3x^3 - 5x^2 + 7x + a$	$x - 2$	21
(4)	$5x^2 + ax + 11$	$x + 2$	43
(5)	$4x^2 + ax + 9$	$x + 3$	66
(6)	$ax^2 - 5x + 13$	$x - 3$	16

(7) If $(x + 2)$ divides $3x^3 + ax^2 - 5x + 2$ leaving no remainder, find the value of a.

(8) If $(x - 2)$ and $(x + 5)$ both divide $x^2 + ax + b$ withuot leaving any remainder, find the polynomial.

(9) $x^2 + ax + b$ leaves a remainder 13, when divided by $(x - 1)$ and a remainder 23, when divided by $(x - 2)$. Find the values of a and b.

(10) The remainder is 21 when $2x^2 + ax + b$ is divided by $(x + 2)$ and 11 when it is divided by $(x - 3)$. Find the values of a and b.

ANSWERS

(1) $a = 7$ (4) $a = -6$ (7) $a = 3$ (10) $a = -4$, $b = 5$.

3-11. Factor Theorem

If a number divides another number without leaving any remainder, the first number is called the divisor of the 2nd number. This applies to polynomials also. We can find the divisors or factors of the polynomials, with the help of the remainder theorem.

Example :

Find whether $(x - 3)$, $(x - 2)$ and $(x - 1)$ may be the factors of $x^2 - 5x + 6$.

Solution :

$P(x) = x^2 - 5x + 6$

If divided by $(x - 3)$, the remainder is

$P(3) = 9 - 15 + 6 = 0.$

If divided by $(x - 2)$, the remainder is

$P(2) = 4 - 10 + 6 = 0.$

If divided by $(x - 1)$, the remainder is

$P(1) = 1 - 5 + 6 = 2.$

$\therefore$ $(x - 3)$ and $(x - 2)$ are the factors of $x^2 - 5x + 6$.

$(x - 1)$ is not a factor.

Factor Theorem:

If $(x - a)$, $a \in R$, is a factor of a polynomial, $P(x)$, then $P(a) = 0$.

Conversely if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$. a is called the **zero** of the **polynomial** $P(x)$.

Exercise 3·11

(1) Examine whether $(x - 1)$ is a factor of the following polynomials.

(a) $x^2 - 4x + 3$ (b) $x^2 + x + 1$ (c) $2x^2 + x - 3$
(d) $x^3 - x^2 + x + 1$ (e) $x^2 - 8x + 7$

(2) Show that $(x - 2)$ is a factor of the following polynomials.

(a) $x^2 + 3x - 10$ (b) $x^2 - 9x + 14$ (c) $2x^2 - 7x + 6$
(d) $x^3 - 8$

(3) Show that $(x + 3)$ is a factor of the following polynomials.

(a) $x^2 - 2x - 15$ (b) $x^2 + 12x + 27$
(c) $2x^2 + 13x + 21$ (d) $7x^3 + 21x^2 - 2x - 6$
(e) $x^3 + 27$

(4) Some polynomials are given below. One of the factors is given against each. Find the value of a.

	Polynomial	Factor
(a)	$2x^2 + ax - 10$	$x - 2$
(b)	$7x^2 - 11x + a$	$x - 1$
(c)	$2x^2 + ax - 3$	$x + 3$
(d)	$3x^3 + ax^2 - 5x + 2$	$x + 2$

(5) If $(x - 3)$ and $(x + 2)$ are the factors of the polynomial $x^3 + px^2 - 2x + q$, find the values of p and q.

ANSWERS

(4) (a) $a = 1$ (d) $a = 3$
(5) $p = -5$; $q = 24$

3-12. Synthetic Division

Let us divide $x^3 + 5x^2 - 4x + 6$ by $x - 2$

$$
\begin{array}{r|l}
x - 2 & x^3 + 5x^2 - 4x + 6 \quad (x^2 + 7x + 10 \\
 & \underline{x^3 - 2x^2} \\
 & \quad\quad 7x^2 - 4x \\
 & \quad\quad \underline{7x^2 - 14x} \\
 & \quad\quad\quad\quad 10x + 6 \\
 & \quad\quad\quad\quad \underline{10x - 20} \\
 & \quad\quad\quad\quad\quad\quad \underline{26}
\end{array}
$$

Let us examine how we got the expressions at each step in this division.

$$5x^2 + 2x^2 = 7x^2$$
$$-4x + 14x = 10x$$
$$6 + 20 = 26$$

When we subtract, we add the additive inverse. Bearing this fact in mind, we can do this division in a shortcut method. (First write the co-efficients and the constant term in order)

$$
\begin{array}{c|cccc}
2 & 1 & 5 & -4 & 6 \\
 & & 2 & 14 & 20 \\
\hline
 & 1 & 7 & 10 & \boxed{26}
\end{array}
\quad \text{Remainder}
$$

Quotient.

Quotient $= x^2 + 7x + 10$

Remainder $= 26$.

Exercise 3-12

Find the quotient and the remainder when the following polynomials are divided by the binomials given against them, using the shortcut method of division.

	Polynomials	Binomials
(1)	$4x^2 - 5x + 13$	$(x - 1)$, $(x - 2)$, $(x - 3)$
(2)	$2x^2 + 7x - 15$	$(x + 1)$, $(x - 2)$, $(x + 3)$

(3) $x^2 - 10x + 8$ $(x - 3)$, $(x + 1)$, $(x - 4)$

(4) $3x^2 - 10x + 11$ $(x - 5)$, $(x + 2)$, $(x + 4)$

(5) $2x^3 - 3x^2 + 5x + 7$ $(x + 5)$, $(x - 4)$, $(x + 3)$

(6) $4x^3 - 12x^2 + 5$ $(x + 1)$, $(x - 2)$, $(x - 4)$

ANSWERS

(1) $4x - 1$, 12 ; $4x + 3$, 19 ; $4x + 7$, 34.

(5) $2x^2 - 13x + 70$, $- 343$; $2x^2 + 5x + 25$, 107.
$2x^2 - 9x + 32$, $- 89$.

3-13. Factorization : Polynomials of Third degree

Let us see the method of finding the factors of the third degree expressions, using the shortcut.

Illustrative Example :

Factorize : $2x^3 - 5x^2 + x + 2$

The divisors of the constant term, namely 2 are 1, -1, 2, -2.

Let us try for which values $P(x) = 0$

1	2	−5	1	2
		2	−3	−2
2	2	−3	−2	0
		4	2	
	2	1	0	

$P(1) = 0$; $P(2) = 0$

$\therefore$ $(x - 1)$ and $(x - 2)$ are factors.

$(2x + 1)$, the quotient is the third factor.

$2x^3 - 5x^2 + x + 2 = (x - 1)(x - 2)(2x + 1)$

Exercise 3-13

Factorize :

(1) $x^3 - 2x^2 - 5x + 6$ (2) $x^3 + 2x^2 - x - 2$

(3) $x^3 - 6x^2 + 11x - 6$ (4) $x^3 - 4x + 3$

(5) $x^3 + 6x^2 + 11x + 6$ (6) $x^3 - x^2 - x - 2$

(7) $x^3 - 6x^2 - x + 6$ (8) $x^3 - 7x - 6$

(9) $x^3 + 2x - 3$ (10) $2x^3 - 3x^2 - 11x + 6$

ANSWERS

(1) $(x-3)(x-1)(x+2)$ (2) $(x+2)(x+1)(x-1)$

(3) $(x-1)(x-2)(x-3)$ (4) $(x-1)(x^2+x-3)$

(5) $(x+1)(x+2)(x+3)$ (6) $(x-2)(x^2+x+1)$

(7) $(x+1)(x-1)(x-6)$ (8) $(x+1)(x+2)(x-3)$

(9) $(x-1)(x^2+x+3)$ (10) $(x+2)(x-3)(2x-1)$

3-14. Polynomials : General expressions

Degree	Variable or Variables	General form	Number of Terms
1	x	$ax+b$	2
2	x	ax^2+bx+c	3
1	x, y	$ax+by+c$	3
2	x, y	$ax^2+by^2+cxy+dx+ey+f$	6
1	x, y, z	$ax+by+cz+d$	4
2	x, y, z	$ax^2+by^2+cz^2+dxy+eyz+fzx+gx+hy+lz+m$	10

Exercise 3-14

(1) Write the general form of the polynomial with the given variable.

(a) A polynomial of the first degree with m as variable.

(b) A 2nd degree polynomial in p and q.

(c) First degree polynomial in p, q, r.

(d) 2nd degree polynomial in p, q, r.

(2) In a polynomial of the first degree in x, if $x = 4$, its value is 11. If $x = 3$, its value is 9. Find the polynomial.

ANSWERS

(2) $2x + 3$.

3-15. Homogeneous Expressions

If every term of a polynomial is of the same degree, the polynomial is said to be homogeneous.

e.g. $3x^3 + 2x^2y + 4xy^2 + 4y^3$

In this expression, every term is of the third degree. So this is a homogeneous expression of the third degree.

Note: (1) A homogeneous expression is so called because of its terms being of the same degree and has nothing to do with the nature of the co-efficients.

(2) There can be no constant term in a homogeneous expression.

General forms of the homogeneous expressons :

Variables	Degree	General form
x, y.	1	$ax + by$
x, y.	2	$ax^2 + bxy + cy^2$
x, y, z.	1	$ax + by + cz$
x, y, z.	2	$ax^2 + by^2 + cz^2 + dxy + eyz + fzx$

Exercise 3-15

1. Write the homogeneous expressions for the given degree in the variables specified.
   (a) 1st degree in p, q.
   (b) 1st degree in a, b, c.
   (c) 2nd degree in p, q, r.

2. What is the degree of the homogeneous expressions.
   $$4x^2 - 5xy + 6y^2 - 4yz + 11z^2$$

3. Is $(x + 3)^2$ a homogeneous expression ? why ?

4. In a homogeneous expression of the first degree in x, y, if $x = 3, y = 1$, its value is 17. If $x = 2$, $y = 3$, its value is 23. Find the expression.

ANSWERS

(4) $4x + 5y$

3-16. Symmetric expressions

Let us take the expression $3x^2 + 7xy + 3y^2$ substitute y for x and x for y. Then the expression becomes $3y^2+7yx+3x^2$. This expression is the same as the first one. This is called a symmetric expression. If an expression in 3 variables, does not change after interchanging any two variables at a time, it is called a symmetric expression. What do you find about the coefficients of terms in symmetric expressions ?

General forms of the symmetric expressions :

Variables	Degree	Expression (polynomial)
x, y.	1	$a(x + y) + b$
x, y.	2	$a(x^2+y^2)+bxy+c(x+y)+d$
x, y, z.	1	$a(x+y+z)+b$
x, y, z	2	$a(x^2+y^2+z^2)+b(xy+yz+zx)+c(x+y+z)+d$

Exercise 3-16

(1) Which of the following expressions are symmetric?

(a) $x^2 + y^2 + 3x + 3y + 5$

(b) $x^5 + y^5 + 4$

(c) $2x^2 - 2y^2$

(d) $3x^2 + 2xy + 3y^2 + 6x + 8 + 6y$

(e) $3x^2 + 3y^2 + x + 7$

(f) $5a^2 - 3ab + 5b^2 - 2a + 3b - 7$

(2) Write a symmetric expression of the 2nd degree in a, b, c.

3-17. Expressions : Symmetric and Cyclic

Let us take $a(b+c) + b(c+a) + c(a+b)$. In this substitute b for a, c for b and a for c. Then the expression becomes

$$b(c + a) + c(a + b) + a(b + c)$$

This is the same as the first expression. Such expressions are called expressions of cyclic symmetry. Note from fig 3·1, how we substitute the variables.

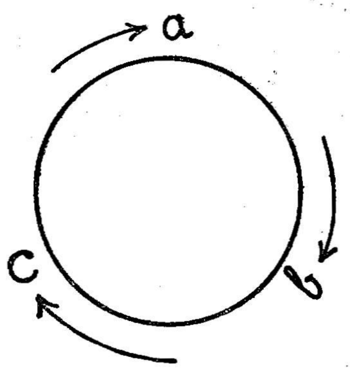


Fig. 3-1.

$a(b+c) + b(c+a) + c(a+b)$ can be written as $\Sigma a(b+c)$ for short.

Σ is pronounced as 'Sigma'.

This is used in the sense of additon. By substituting we get the other two terms in the cyclic expression.

Example 1 :

Expand and simplify : $\Sigma\, x\,(y - z)$

Solution :

$$\Sigma\, x\,(y - z) = x\,(y - z) + y\,(z - x) + z\,(x - y)$$
$$= xy - xz + yz - yx + zx - zy$$
$$= 0.$$

Example 2 :

$\Sigma\, x^2 + 2\,\Sigma\, xy = ?$

Solution :

$$\Sigma\, x^2 + 2\,\Sigma\, xy$$
$$= x^2 + y^2 + z^2 + 2\,(xy + yz + zx)$$
$$= x^2 + y^2 + z^2 + 2\,xy + 2\,yz + 2\,zx$$
$$= (x + y + z)^2$$

Exercise 3-17

Write the expansions and simplify each of the following:

(1) $\Sigma\, x$ (2) $\Sigma\, a$

(3) $\Sigma\, (a + b)$ (4) $\Sigma\, x\,(y + z)$

(5) $\Sigma\, (a^2 + b^2)$ (6) $\Sigma\, (x^3 + y^3)$

(7) $\Sigma\, (x + y)\,(x - y)$ (8) $\Sigma\, (a + b + c)$

(9) $\Sigma\, (a + b - c)$ (10) $\Sigma\, (a - b - c)$

(11) $\Sigma\, (x-y)\,(x^2+xy+y)$ (12) $\Sigma\, (a^2 + b^2 + c^2)$

(13) $\Sigma\, (x^2 + y^2 - z^2)$ (14) $(\Sigma\, x)^2 - \Sigma\, x^2$

(15) $\Sigma\, p^2\,(q + r - p)$

ANSWERS

(1) $x + y + z$ (3) $2a + 2b + 2c$

(4) $2\,(xy + yz + zx)$ (10) $-\,(a + b + c)$

3-18. Polynomials : Square root

Recall :

$$x \times x = x^2$$

$$(-x) \times (-x) = x^2$$

$$\sqrt{x^2} = +x \text{ or } -x.$$

We write this as $\sqrt{x^2} = \pm x$

$$x^3 \times x^3 = (-x^3) \times (-x^3) = x^6$$

$$\sqrt{x^6} = \pm x^3 = \pm x^{6 \times \frac{1}{2}}$$

$$x^5 \times x^5 = (-x^5) \times (-x^5) = x^{10}$$

$$\sqrt{x^{10}} = \pm x^5 = \pm x^{10 \times \frac{1}{2}}$$

$$5x^4 \times 5x^4 = (-5x^4) \times (-5x^4) = 25x^3$$

$$\sqrt{25x^8} = \pm 5x^4 = \pm 5x^{8 \times \frac{1}{2}}$$

Remember :

(1) Find the square root of the numerical co-efficient of the terms as in Arithmetic.

(2) Find the power of the base. The power of the square root is half the power of the base.

(3) Every number will have two square roots, one positive and the other negative.

We generally express the positive square root only, unless specifically stated.

Example :

Find the square root of $144a^{12} b^{18} c^2$

Solution :

$$\sqrt{144a^{12} b^{18} c^2}$$

$$= 12a^{12 \times \frac{1}{2}} b^{8 \times \frac{1}{2}} c^{2 \times \frac{1}{2}}$$

$$= 12a^6 b^4 c^1$$

$$= 12a^6 b^4 c.$$

Exercise 3-18

Find the square roots of the following.

(Mental sums)

(1) a^{10} (2) x^{16} (3) $25x^6$ (4) $169x^4y^8$

(5) $2{\cdot}25\,x^8y^2$ (6) $\dfrac{9a^4b^4}{16x^2y^2}$ (7) $\dfrac{8m^8n^6}{50a^2b^{12}}$

(8) $\dfrac{0{\cdot}81m^2\,(x+y)^2}{1{\cdot}21\,p^4\,(q+r)^6}$ (9) $1\dfrac{9}{16}x^{10}\,y^{12}\,z^{14}$

(10) $36\,(x+y)^4\,(y+z)^6\,(z+x)^2$.

3-19. Square root : Method of Factorisation

Recall :

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(2x+3y)^2 = 4x^2 + 12xy + 9y^2$$
$$(3x-4y)^2 = 9x^2 - 24xy + 16y^2$$

$$\sqrt{a^2+2ab+b^2} = (a+b)$$
$$\sqrt{a^2-2ab+b^2} = (a-b)$$
$$\sqrt{4x^2+12xy+9y^2} = (2x+3y)$$
$$\sqrt{9x^2-24xy+16y^2} = (3x-4y)$$

Example 1 :

Find the square root of $25x^2 + 30xy + 9y^2$.

Solution :

$$25x^2 + 30xy + 9y^2$$
$$= (5x)^2 + 2(5x)(3y) + (3y)^2$$
$$= (5x+3y)^2$$
$$\therefore\ \sqrt{25x^2+30xy+9y^2} = (5x+3y)$$

Example 2 :

Find the square root of $(x+y)^2 - 6\,(x+y) + 9$.

Solution :

$(x + y)^2 - 6(x + y) + 9$

Let $x + y$ be k.

$$(x+y)^2 - 6(x+y) + 9 = k^2 - 6k + 9$$
$$= k^2 - 2(k)(3) + 3^2$$
$$= (k - 3)^2$$

$$\text{Square root} = (k - 3)$$
$$= (x + y - 3).$$

Exercise 3-19

Find the square roots of the following.

(1) $a^2 + 2a + 1$

(2) $x^2 + 14x + 49$

(3) $x^2 + 16x + 64$

(4) $b^2 - 10b + 25$

(5) $c^2 - 20c + 100$

(6) $m^2 - 22m + 121$

(7) $16x^2 + 24x + 9$

(8) $9x^2 + 30x + 25$

(9) $49x^2 - 14x + 1$

(10) $36a^2 - 60a + 25$

(11) $9x^2 + 48xy + 64y^2$

(12) $16a^2 + 72ab + 81b^2$

(13) $4l^2 - 28lm + 49m^2$

(14) $25c^2 - 60cd + 36d^2$

(15) $4a^2b^2 + 4ab + 1$

(16) $25x^2y - 20xy + 4$

(17) $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

(18) $p^2 + q^2 + r^2 + 2pq - 2qr + 2rp$

(19) $(m + n)^2 + 12(m + n) + 36$

(20) $(x + y)^2 - 10(x + y) + 25$

(21) $16(x - y)^2 + 8(x - y) + 1$

(22) $4(p + q)^2 - 28(p + q) + 49$

ANSWERS

(1) $(a + 1)$ (4) $(b - 5)$ (7) $(4x + 3)$

(11) $(3x + 8y)$ (15) $(2ab + 1)$ (17) $(x + y + z)$

(19) $(m + n + 6)$ (22) $(2p + 2q - 7)$

3·20. Square root : Product of expressions:

Find the square root of :

$$(x^2 + 2x - 15)(x^2 - x - 6)(x^2 + 7x + 10)$$

Let us first find the factors of the given expressions.

$$x^2 + 2x - 15 = (x + 5)(x - 3)$$

$$x^2 - x - 6 = (x - 3)(x + 2)$$

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

$$\sqrt{(x^2 + 2x - 15)(x^2 - x - 6)(x^2 + 7x + 10)}$$

$$= \sqrt{(x + 5)(x - 3)(x - 3)(x + 2)(x + 2)(x + 5)}$$

$$= \sqrt{(x + 5)^2 \cdot (x - 3)^2 \; (x + 2)^2}$$

$$= (x + 5)(x - 3)(x + 2)$$

Exercise. 3 20

Find the square root of the following :

(1) $(x^2 + 3x + 2)(x^2 + 5x + 6)(x^2 + 4x + 3)$

(2) $(a^2 + 7a + 12)(a^2 + 9a + 20)(a^2 + 8a + 15)$

(3) $(p^3 - 3p + 2)(p^2 - 5p + 6)(p^2 - 4p + 3)$

(4) $(m^2 - 7m + 12)(m^3 - 9m + 20)(m^2 - 8m + 15)$

(5) $(x^2 - 3x - 4)(x^2 + 3x - 28)(x^2 + 8x + 7)$

(6) $(x^2 + 4x - 5)(x^2 + 11x + 30)(x^2 + 5x - 6)$

(7) $(x^2 + 7x - 18)(x^2 - 5x + 6)(x^2 + 6x - 27)$

(8) $(a^2 - 9)(a^2 + 2a - 15)(a^2 + 8a + 15)$

(9) $(12x^2 + 19x + 5)(8x^2 + 22x + 15)(6x^2 + 11x + 3)$

(10) $(4x^2 - 8x + 3)(4x^2 - 12x + 5)(4x^2 - 16x + 15)$

ANSWERS

(1) $(x + 1)(x + 2)(x + 3)$

(2) $(a + 3)(a + 4)(a + 5)$

(3) $(p - 1)(p - 2)(p - 3)$

(4) $(m - 3)(m - 4)(m - 5)$

(5) $(x + 1)(x - 4)(x + 7)$

(6) $(x - 1)(x + 5)(x + 6)$

(7) $(x - 2)(x - 3)(x + 9)$

(8) $(a + 3)(a - 3)(a + 5)$

(9) $(3x + 1)(2x + 3)(4x + 5)$

(10) $(2x - 1)(2x - 3)(2x - 5)$

3-21. Square root : Division method

We can use the division method in finding the square root of polynomials, as in finding the square root of numbers in Arithmetic.

Let us find the square root of $9x^2 - 30x + 25$.

$$\begin{aligned} \text{Let } 9x^2 - 30x + 25 \text{ be } &= (3x + a)^2 \\ &= 9x^2 + 6ax + a^2 \\ -30x + 25 &= 6ax + a^2 \\ &= a(6x + a) \\ &= a(2 \times 3x + a) \end{aligned}$$

We know that $a = -5$

We can show this briefly in the division method.

	$3x - 5$	
$3x$	$9x^2 - 30x + 25$ $9x^2$	$(3x)^2$
$6x - 5$	$-30x + 25$ $-30x + 25$	$(6x - 5)(-5)$
	0	

Example :

Find the square root of $4x^4 - 12x^3 + 29x^2 - 30x + 25$.

Solution :

	$2x^2-3x+5$	
$2x^2$	$4x^4-12x^3+29x^2-30x+25$	
	$4x^4$	$(2x^2)^2$
$4x^2-3x$	$-12x^3+29x^2$	$(4x^2-3x)\times(-3x)$
	$-12x^3+9x^2$	$\frac{-12x^3}{4x^2}=-3x$
$4x^2-6x+5$	$20x^2-30x+25$	$(4x^2-6x+5)\times 5$
	$20x^2-30x+25$	$\frac{20x^2}{4x^2}=5$
	0	

$$\therefore \quad \sqrt{4x^4-12x^3+29x^2-30x+25} = 2x^2-3x+5$$

Exercise 3-21

Find the square root of the following polynomials.

(1) $x^4 + 2x^3 + 3x^2 + 2x + 1$

(2) $x^4 - 4x^3 + 6x^2 - 4x + 1$

(3) $4x^4 - 20x^3 + 29x^2 - 10x + 1$

(4) $25p^4 - 20p^3 - 26p^2 + 12p + 9$

(5) $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$

(6) $9x^6 - 24x^5 + 16x^4 - 30x^3 + 40x^2 + 25$

(7) If $16x^4 - 24x^3 - 7x^2 + ax + 4$ is a perfect square, find the value of a.

(8) If $25x^4 - 30x^3 - x^2 + ax + b$ is a perfect square, find the values of a and b.

ANSWERS

(1) $(x^2 + x + 1)$ (2) $(x^2 - 2x + 1)$

(3) $(2x^2 - 5x + 1)$ (4) $(5p^2 - 2p - 3)$

(5) $(x^3 + x^2 + x + 1)$ (6) $(3x^3 - 4x^2 - 5)$

(7) $a = 12$ (8) $a = 6,\ b = 1$

3.22. Greatest common divisor (g. c. d.)

Recall :

We have already learnt how to find the greatest common divisor of 24, 30 and 36.

$$24 = 2^3 \times 3$$
$$30 = 2 \times 3 \times 5$$
$$36 = 2^2 \times 3^2$$

The g. c. d. of 24, 30 and 36 = $2 \times 3 = 6$. Similarly the g. c. d. of $5a^2b^2$, $20ab^3$ can be found out.

$$5a^2b^2 = 5 \times a^2 \times b^2$$
$$20ab^3 = 5 \times 2^2 \times a \times b^3$$
$$\therefore \text{ g. c. d.} = 5ab^2$$

Example :

Find the g. c. d. of $4x^2y^3$ and $6xy^5$.

Solution :

$$4x^2y^3 = 2^2 \times x^2 \times y^3$$
$$6xy^5 = 2 \times 3 \times x \times y^5$$
$$\therefore \text{ g. c. d.} = 2 \times x \times y^3$$
$$= 2xy^3$$

Exercise 3-22

Find the g. c. d. of the following :

(1) a^7, a^4

(2) x^5, x^6

(3) p^8, p^{11}, p^7

(4) $4a^3$, $6a^4$

(5) $7x^3$, $14x^5$

(6) $24a^3$, $36a^7$

(7) $10a^2b^2$, $15a^4b^3$

(8) $20x^3y^3$, $25xy^4$

(9) $12a^4b^2c^6$, $16ab^3c^4$

(10) $12a^2b^2c^2$, $30a^3bc^2$

(11) $10a^2bc^2$, $15ab^2c^3$, $20b^3c^2$

(12) $4x^2y^3z^2$, $12x^5$

(13) $2xb$, axy

(14) p^m, p^{m+1}, p^{m+2}

ANSWERS

(1) a^4 (4) $2a^3$ (7) $5a^2b^2$ (11) $5bc^2$ (14) p^m

3-23. Greatest Common Divisor: Three expressions

To find the g. c. d. of more than two expressions, first find their factors. Then find their common factors. The greatest of them is the g. c. d. of those expressions.

Example :

Find the g. c. d. of

$x^2 - 3x - 10,\ x^2 + 9x + 14$ and $x^2 - x - 6$

Solution :

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$
$$x^2 + 9x + 14 = (x + 7)(x + 2)$$
$$x^2 - x - 6 = (x - 3)(x + 2)$$
$$\text{g. c. d.} = (x + 2)$$

Exercise 3·23

Find the g. c. d. of the following :

(1) $(x + 3)(x - 5)(x + 7),\ (x - 2)(x + 3)(x + 6)$

(2) $(a - 1)(a - 2)(a - 3),\ (a + 5)(a - 1)(a - 2)$

(3) $(a - b),\ (a - b)^3$

(4) $x^2(x - y),\ x^3(x - y)^2$

(5) $3x - 5,\ 9x - 15$

(6) $x^2 + xy,\ x^2 - y^2$

(7) $a^2 - b^2,\ a^3 - b^3$

(8) $x + y,\ x^3 + y^3$

(9) $p^2 - 4,\ p^2 + 3p + 2,\ p^2 + p - 2$

(10) $x^2 - 6x + 9,\ x^3 - 27$

(11) $x^2 + 8x + 15,\ x^2 + 4x - 5,\ x^2 + 3x - 10$

(12) $6a^2 - 7a - 3,\ 10a^2 - 11a - 6,\ 2a^2 + 11a - 21$

(13) $x^2 - 6x - 7,\ 2x^2 - x - 3,\ x^2 - 1$

(14) $m^2 + 2mn + n^2,\ m^2 - n^2,\ (m + n)^3$

ANSWERS

(1) $(x + 3)$ (4) $x^2(x - y)$ (8) $x + y$

(11) $(x + 5)$ (12) $(2a - 3)$

3-24. Least Common Multiple (l. c. m.)

Let us recall how to find the l. c. m. of 25 and 40

$$25 = 5^2$$
$$40 = 5 \times 2^3$$
$$\text{l. c. m.} = 5^2 \times 2^3$$
$$= 25 \times 8$$
$$= 200$$

Example :

Find the l. c. m. of $16x^2$, $24x^3y$

Solution :

$$16x^2 = 2^4 \times x^2$$
$$24x^3y = 2^3 \times 3 \times x^3 \times y$$
$$\text{l. c. m.} = 2^4 \times 3 \times x^3 \times y$$
$$= 48x^3y$$

Exercise 3-24

Find the l. c. m. of the following :

(1) x^2, x^3 (2) a^4, a^6 (3) x^6, x^{11}, x^3

(4) $2a^2, 3a^5$ (5) $8x^4, 12x^3$ (6) $12x^3, 20x^5$

(7) xy, yz (8) a^2b, ab^2 (9) x^2y^3, xy^3

(10) $3ab, 6bc, 9ca$ (11) x^2yz, x^3y

(12) $10a^2bc, 15abc^2, 20a^2b^2c$

ANSWERS

(1) x^3 (4) $6a^5$ (9) x^2y^3 (11) x^3yz (12) 60 $a^2b^2c^2$

3-25. Least Common Multiple : Three expressions

Find the l. c. m. of

$$x^2 + 2x - 3,\ x^2 - 6x + 8,\ x^2 + x - 6$$

Solution :

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$
$$x^2 - 6x + 8 = (x - 4)(x - 2)$$
$$x^2 + x - 6 = (x + 3)(x - 2)$$
$$\text{l. c. m.} = (x + 3)(x - 2)(x - 1)(x - 4)$$

Exercise 3-25

Find the l. c. m. of the following :

(1) $(x + y), (x + y)^2, (x + y)^3$

(2) $2(x - y), 4(x - y)^2, 6(x - y)^3$

(3) $10(p - q)(q - r), 15(q - r)(r - p)$

(4) $x^2y(x + y), 2x^2z(x - y)$

(5) $m + n, m^2 - n^2$

(6) $m + n, m^3 + n^3$

(7) $a^2 + 2ab + b^2, a^2 - b^2$

(8) $x^3 - 1, x^2 - 1, (x - 1)^2$

(9) $x^2 + 3x + 2, x^2 + 5x + 6, x^2 + 7x + 12$

(10) $x^2 - 9, x^2 + 2x - 15, x^2 + 7x + 10$

(11) $2a^2 - 5a - 12, a^2 + a - 20, 4a^2 + 8a + 3$

(12) $6x^2 + 13x + 6, 15x^2 + 7x - 2, 10x^2 + 13x - 3$

ANSWERS

(1) $(x + y)^3$ (4) $2x^2yz(x^2 - y^2)$

(8) $(x - 1)^2(x + 1)(x^2 + x + 1)$

(11) $(2a + 3)(a - 4)(a + 5)(2a + 1)$

3-26. Rational Expressions : Simplifying

Recall :

3, −5, 4, −7 are integers.

$\frac{3}{4}, \frac{3}{-5}, \frac{3}{-7}, \frac{4}{-7}$ are rational numbers.

Similarly if $P(x)$ is a polynomial and $Q(x) \neq 0$ is another polynomial, $\frac{P(x)}{Q(x)}$ is called a rational expression.

$\frac{x^2 - 5}{x^2 - 4x + 7}, \frac{x^2 - 3x - 3}{x^3 - 2x^2 + 5x + 1}$ are some examples of rational expressions.

As in Arithmetic, we can add, subtract, multiply and divide the rational expressions. The same rules are applied here.

Example : $\frac{6}{10} = \frac{2 \times 3}{2 \times 5} = \frac{3}{5}$

Similarly, $\frac{4x - 20}{3x - 15} = \frac{4(x-5)}{3(x-5)} = \frac{4}{3}$

$$\frac{2x - 20}{3x^2 - 30x} = \frac{2(x-10)}{3x(x-10)} = \frac{2}{3x}$$

$$\frac{x^2 - 3x + 2}{x^2 - 4x + 3} = \frac{(x-2)(x-1)}{(x-3)(x-1)} = \frac{(x-2)}{(x-3)}$$

Exercise 3-26

Simplify :

(1) $\frac{x^2}{x^2 + xy}$ (2) $\frac{4x}{2x^2 - 2xy}$

(3) $\frac{5x + 10}{5x - 15}$ (4) $\frac{3a - 12b}{3a + 3b}$

(5) $\frac{x - 4}{4 - x}$ (6) $\frac{2a - 6}{3 - a}$

(7) $\frac{x + y}{x^2 - y^2}$ (8) $\frac{6(x+y)^3}{30(x+y)^5}$

(9) $\frac{x^2 + 11x + 30}{x^2 + 2x - 15}$ (10) $\frac{x^2 + x - 2}{x^2 - 1}$

(11) $\frac{x^2 + 2x + 1}{x^2 + 3x + 2}$ (12) $\frac{d^2 - 5d + 4}{d^2 - 10d + 24}$

(13) $\frac{2x^2 + 5x - 12}{4x - 6}$ (14) $\frac{5x - 5}{4x^2 - 9x + 5}$

(15) $\frac{m^3 + 8}{m^2 - m - 6}$ (16) $\frac{4x^2 + 7x - 15}{8x^2 + 2x - 15}$

ANSWERS

(1) $\frac{1}{x + y}$ (5) -1 (8) $\frac{1}{5(x+y)^2}$

(11) $\frac{x + 1}{x + 2}$ (13) $\frac{x + 4}{2}$ (16) $\frac{x + 3}{2x + 3}$

3-27. Rational expressions : Multiplication

$$\frac{4}{7} \times \frac{5}{9} = \frac{4 \times 5}{7 \times 9} = \frac{20}{63}$$

$$\frac{P(x)}{Q(x)} \times \frac{R(x)}{S(x)} = \frac{P(x) \,.\, R(x)}{Q(x) \,.\, S(x)}$$

$$Q(x) \neq 0, \; S(x) \neq 0$$

Example 1 :

Multiply $\frac{2}{x+3}$ by $\frac{4}{x+4}$

Solution :

$$\frac{2}{x+3} \times \frac{4}{x+5} = \frac{2 \times 4}{(x+3)\,(x+5)}$$

$$= \frac{8}{(x+3)\,(x+5)}$$

$$= \frac{8}{x^2 + 8x + 15}$$

Example 2 :

Multiply :

$$\frac{x^2 - x - 6}{x^2 + 4x + 4} \times \frac{x^2 - 2x - 8}{x^2 - 7x + 12}$$

Solution :

$$\frac{x^2 - x - 6}{x^2 + 4x + 4} \times \frac{x^2 - 2x - 8}{x^2 - 7x + 12}$$

$$= \frac{(x-3)\,(x+2)}{(x+2)\,(x+2)} \times \frac{(x-4)\,(x+2)}{(x-4)\,(x-3)}$$

$$= \frac{(x-3)\,(x+2)\,(x-4)\,(x+2)}{(x+2)\,(x+2)\,(x-4)\,(x-5)}$$

$$= 1.$$

Exercise 3-27

Multiply and give the answer, if possible, in a **simplified form.**

(1) $\frac{x-2}{4} \times \frac{-8}{3x-6}$

(2) $\frac{5x+15}{2y-8} \times \frac{3y-12}{4x+12}$

(3) $(x+3) \times \dfrac{3x}{x^2-9}$

(4) $\dfrac{x^2+3x+2}{x+3} \times \dfrac{x+2}{x^2+4x+3}$

(5) $\dfrac{x^2+11x+30}{x^2-6x+9} \times \dfrac{x^2-3x}{x^2-5x}$

(6) $\dfrac{x^2-4x+3}{x^2-9} \times \dfrac{x^2-2x-15}{x^2+6x+5}$

(7) $\dfrac{x^2+3x+2}{x^2-4x-12} \times \dfrac{x^2-7x+6}{x^2-4}$

(8) $\dfrac{x^2+4x+3}{x^2+5x+6} \times \dfrac{x^2+6x+8}{x^2+5x+4}$

(9) $\dfrac{a^2+ab}{a^2-b^2} \times \dfrac{a^3-b^3}{ab(a+b)}$

(10) $\dfrac{3x^2-2x-8}{4x^2+5x-6} \times \dfrac{8x^2+6x-9}{6x^2+17x+12}$

ANSWERS

(1) $\dfrac{-2}{3}$ (5) $\dfrac{x-6}{x-3}$ (7) $\dfrac{x^2-1}{x^2-4}$ (10) $\dfrac{x-2}{x+2}$

3·28. Rational expressions : Division

Recall :

$$\frac{3}{8} \div \frac{15}{16} = \frac{3}{8} \times \frac{16}{15} = \frac{48}{120} = \frac{2}{5}$$

Similarly,

$$\frac{P(x)}{Q(x)} \div \frac{R(x)}{S(x)} = \frac{P(x)}{Q(x)} \times \frac{S(x)}{R(x)}$$

Example :

Simplify : $\dfrac{ax+bx}{a^2+ab+b^2} \div \dfrac{a^2-b^2}{a^3-b^3}$

Solution :

$$\frac{ax+bx}{a^2+ab+b^2} \div \frac{a^2-b^2}{a^3-b^3}$$

$$= \frac{ax+bx}{a^2+ab+b^2} \times \frac{a^3-b^3}{a^2-b^2}$$

$$= \frac{x(a+b)}{a^2+ab+b^2} \times \frac{(a-b)(a^2+ab+b^2)}{(a+b)(a-b)}$$

$$= x$$

Exercise 3-28

Simplify :

(1) $\frac{P+5}{2} \div (3p+15)$

(2) $\frac{3x+12}{5x} \div x^2-16$

(3) $\frac{x-7x+12}{15x^3} \div \frac{x-4}{10x}$

(4) $\frac{x^2-25}{x^2+7x+10} \div \frac{x^2+6x-55}{x^2+6x+8}$

(5) $\frac{x^2-x-20}{x^2+5x+6} \div \frac{x^2+5x+4}{x^2-x-12}$

(6) $\frac{a^2-3ab+2b^2}{a^2-9ab+20b^2} \div \frac{a^2-5ab+6b^2}{a^2-7ab+12b^2}$

(7) $\frac{x^2-16}{x^3-8} \times \frac{2x^2-3x-2}{2x^2+9x+14} \div \frac{3x^2-11x-4}{x^2+2x+4}$

ANSWERS

(1) $\frac{1}{6}$ (3) $\frac{2}{3x^2}(x-3)$ (5) $\frac{(x-5)(x-4)}{(x+2)(x+1)}$ (7) $\frac{1}{3x+1}$

3-29. Rational expressions : Addition : (Same denominators)

Recall : $\frac{4}{11} + \frac{5}{11}$

$$= \frac{4+5}{11}$$

$$= \frac{9}{11}$$

When two rational expressions with the same denominators are added, the numerator is the sum of the numerators and the denominator is the common denominator.

Example 1 :

Add : $\frac{x}{7} + \frac{y}{7}$

Solution :

$$\frac{x}{7} + \frac{y}{7}$$

$$= \frac{x + y}{7}.$$

Example 2 :

Add : $\frac{2x + 5}{x^2 + 5x + 6} + \frac{x + 4}{x^2 + 5x + 6}$

Solution :

$$\frac{2x + 5}{x^2 + 5x + 6} + \frac{x + 4}{x^2 + 5x + 6}$$

$$= \frac{2x + 5 + x + 4}{x^2 + 5x + 6}$$

$$= \frac{3x + 9}{x^2 + 5x + 6}$$

$$= \frac{3(x + 3)}{(x + 3)(x + 2)} = \frac{3}{x + 2}$$

Exercise 3-29

Add : (1) $\frac{x}{7} + \frac{3x}{7}$ (2) $\frac{P}{3} + \frac{q}{3}$

(3) $\frac{4x}{9} + \frac{x}{9}$ (4) $\frac{7x}{24} + \frac{5x}{24}$

(5) $\frac{2}{3x} + \frac{4}{3x}$ (6) $\frac{x - 5}{xy} + \frac{x + 5}{xy}$

(7) $\frac{11x}{9ab} + \frac{7x}{9ab}$ (8) $\frac{5x}{21} + \frac{8x}{21} + \frac{x}{21}$

(9) $\frac{a - 4}{ab} + \frac{a - 6}{ab} + \frac{a + 10}{ab}$

(10) $\dfrac{3}{5x-10}+\dfrac{2}{5x-10}$

(11) $\dfrac{5x-15}{3x-9}+\dfrac{3-x}{3x-9}$

(12) $\dfrac{x}{x^2-36}+\dfrac{6}{x^2-36}$

(13) $\dfrac{x}{x^2+2x+1}+\dfrac{1}{x^2+2x+1}$

(14) $\dfrac{2x}{x^2+10x+24}+\dfrac{12}{x^2+10x+24}$

(15) $\dfrac{2x-10}{x^2-5x+6}+\dfrac{x+1}{x^2-5x+6}$

(16) $\dfrac{x^2+2x}{x^2+6x+9}+\dfrac{2x+3}{x^2+6x+9}$

(17) $\dfrac{x^2+2x}{x^2-7x+12}+\dfrac{3x-36}{x^2-7x+12}$

(18) $\dfrac{x^2}{x^2+10x+21}+\dfrac{6x-7}{x^2+10x+21}$

ANSWERS

(1) $\dfrac{4x}{7}$ (6) $\dfrac{2}{y}$ (9) $\dfrac{3}{b}$ (12) $\dfrac{1}{x-6}$

(15) $\dfrac{3}{x-2}$ (18) $\dfrac{x-1}{x+3}$

3-30. Rational expressions: Addition : (Different denominators)

Recall :

$$\frac{5}{12}+\frac{7}{15}$$

$$=\frac{25}{60}+\frac{28}{60} \text{ (l.c.m. of 12 and 15 is 60)}$$

$$=\frac{53}{60}.$$

Example 1 :

Add : $\dfrac{2}{1+x}+\dfrac{3}{1-x}$

Solution :

l.c.m. of $(1 + x)$ and $(1 - x)$ is $(1 + x)(1 - x)$

$$\frac{2}{1+x} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2-2x}{(1+x)(1-x)}$$

$$\frac{3}{1-x} = \frac{3(1+x)}{(1-x)(1+x)} = \frac{3+3x}{(1+x)(0-x)}$$

$$\frac{2}{1+x} + \frac{3}{1-x} = \frac{2-2x}{(1+x)(1-x)} + \frac{3+3x}{(1+x)(1-x)}$$

$$= \frac{2-2x+3+3x}{(1+x)(1-x)}$$

$$= \frac{5+x}{1-x^2}$$

Example 2 :

Add: $\dfrac{x-4}{x^2-3x+2} + \dfrac{x-2}{x^2-5x+4}$

Solution :

$$x^2 - 3x + 2 = (x-2)(x-1)$$

$$x^2 - 5x + 4 = (x-4)(x-1)$$

The l.c.m. of $x^2 - 3x + 2$ and $x^2 - 5x + 4$

is $(x-1)(x-2)(x-4)$

$$\frac{x-4}{x^2-3x+2} = \frac{x-4}{(x-2)(x-1)} = \frac{(x-4)(x-4)}{(x-2)(x-1)(x-4)}$$

$$= \frac{(x-4)^2}{(x-2)(x-1)(x-4)}$$

$$\frac{x-2}{x^2-5x+4} = \frac{(x-2)}{(x-4)(x-1)} = \frac{(x-2)(x-2)}{(x-4)(x-1)(x-2)}$$

$$= \frac{(x-2)^2}{(x-4)(x-1)(x-2)}$$

$$\frac{x-4}{x^2-3x+2} + \frac{x-2}{x^2-5x+4}$$

$$= \frac{(x-4)^2}{(x-2)(x-1)(x-4)} + \frac{(x-2)^2}{(x-4)(x-1)(x-2)}$$

$$= \frac{x^2 - 8x + 16 + x^2 - 4x + 4}{(x-2)(x-1)(x-4)}$$

$$= \frac{2x^2 - 12x + 20}{(x-1)(x-2)(x-4)}$$

Exercise 3-30

Simplify :

(1) $\frac{x}{2} + \frac{x}{8}$ (2) $\frac{2x}{3} + \frac{7x}{6}$

(3) $\frac{4a}{5} + \frac{6a}{7}$ (4) $\frac{1}{a} + \frac{1}{b}$

(5) $\frac{4}{xy} + \frac{3}{y}$ (6) $\frac{3x}{10} + \frac{2x-3}{15}$

(7) $\frac{3x}{4x+16} + \frac{2x}{5x+20}$ (8) $\frac{3x}{x^2-9} + \frac{5}{x+3}$

(9) $\frac{2a+3}{a^2-16} + \frac{4a}{3a+12}$

(10) $\frac{x+3}{x^2-8x+15} + \frac{2}{3x-9}$

(11) $\frac{2x-1}{x^2+6x+8} + \frac{5}{2x+4}$

(12) $\frac{2x-1}{x^2-25} + \frac{x+2}{x+5}$

(13) $\frac{3x}{x^2+3x+2} + \frac{4x}{x^2+7x+10}$

(14) $\frac{x+1}{3x^2-2x-1} + \frac{x-1}{3x^2+4x+1}$

(15) $\frac{1}{x^2+4x+3} + \frac{1}{x^2+5x+6} + \frac{1}{x^2+3x+2}$

(16) $5 + \frac{2}{2x-3}$ (17) $7 + \frac{x+1}{x-2}$

ANSWERS

(1) $\frac{5x}{8}$ (5) $\frac{4+3x}{xy}$ (8) $\frac{8x-15}{x^2-9}$ (11) $\frac{9}{2(x+4)}$

(14) $\frac{2x^2+2}{(3x+1)(x+1)(x-1)}$ (17) $\frac{8x-13}{x-2}$

3-31. Rational expressions : Subtraction

Example 1 :

Simplify :

$$\frac{2x}{x+3} - \frac{x}{4}$$

Solution :

l.c.m. of 4 and $(x + 3)$ is $4\,(x + 3)$

$$\frac{2x}{x+3} = \frac{2x\,(4)}{(x+3)\,4} = \frac{8x}{4\,(x+3)}$$

$$\frac{x}{4} = \frac{x\,(x+3)}{4\,(x+3)} = \frac{x^2+3x}{4\,(x+3)}$$

$$\frac{2x}{x+3} - \frac{x}{4} = \frac{8x}{4\,(x+3)} - \frac{x^2+3x}{4\,(x+3)}$$

$$= \frac{8x - x^2 - 3x}{4\,(x+3)}$$

$$= \frac{-x^2+5x}{4\,(x+3)}$$

Example 2 :

Simplify :

$$\frac{x+1}{x^2+7x+12} - \frac{x+2}{x^2+8x+15}$$

Solution :

$$x^2 + 7x + 12 = (x+3)\,(x+4)$$

$$x^2 + 8x + 15 = (x+3)\,(x+5)$$

l.c.m. of $x^2 + 7x + 12$, $x^2 + 8x + 15$ is $(x+3)\,(x+4)\,(x+5)$

$$\frac{x+1}{x^2+7x+12} = \frac{x+1}{(x+3)\,(x+4)} = \frac{(x+1)\,(x+5)}{(x+3)\,(x+4)\,(x+5)}$$

$$\frac{x+2}{x^2+8x+15} = \frac{x+2}{(x+3)\,(x+5)} = \frac{(x+2)\,(x+4)}{(x+3)\,(x+4)\,(x+5)}$$

$$\frac{x+1}{x^2+7x+12} - \frac{x+2}{x^2+8x+15} = \frac{(x+1)\,(x+5)}{(x+3)\,(x+4)\,(x+5)} - \frac{(x+2)\,(x+4)}{(x+3)\,(x+4)\,(x+5)}$$

$$= \frac{(x + 1)(x + 5) - (x + 2)(x + 4)}{(x + 3)(x + 4)(x + 5)}$$

$$= \frac{x^2 + 6x + 5 - x^2 - 6x - 8}{(x + 3)(x + 4)(x + 5)}$$

$$= \frac{-3}{(x + 3)(x + 4)(x + 5)}$$

Exercise 3-31

Simplify :

(1) $\frac{4a}{7} - \frac{a}{7}$ (2) $\frac{3x}{8} - \frac{x}{8}$

(3) $\frac{7x}{12} - \frac{3x}{8}$ (4) $\frac{5}{ab} - \frac{3}{b}$

(5) $\frac{4}{x^2 y} - \frac{3}{xy^2}$ (6) $\frac{3}{2p^2q^2} - \frac{5}{6pq}$

(7) $\frac{a}{a + 3} - \frac{2}{a - 3}$ (8) $\frac{7}{x - 5} - \frac{3}{5 - x}$

(9) $\frac{5x}{x^2 - 9} - \frac{2}{x + 3}$ (10) $\frac{7}{2x - 10} - \frac{5x}{x^2 - 25}$

(11) $\frac{2x + 1}{x^2 - 36} - \frac{x + 7}{x + 6}$.

(12) $\frac{2p}{3p^2 + 5p - 2} - \frac{p}{3p^2 - 7p + 2}$

(13) $\frac{4x - 5}{x^2 - 7x + 12} - \frac{x + 7}{x^2 - 7x + 12}$

(14) $\frac{5}{x^2 + 5x + 6} - \frac{4}{x^2 + 5x + 6}$

(15) $\frac{x}{x^2 - 3x + 2} + \frac{x}{x^2 - 5x + 6} - \frac{2x}{x^2 - 4x + 3}$

ANSWERS

(1) $\frac{3a}{7}$ (3) $\frac{5x}{24}$

(6) $\frac{9 - 5pq}{6p^2q^2}$ (10) $\frac{-3x + 35}{2(x^2 - 25)}$

(12) $\frac{p^2 - 6p}{(3p - 1)(p + 2)(p - 2)}$ (15) 0.

3-32. Quadratic equations

We know already how to solve the simple equation, $ax + b = 0,\ x \in R$.

If the degree of the variable in an equation is 2, then it is called a quadratic equation.

$ax^2 + bx + c = 0$; $a, b, c \in R$, $a \neq 0$ is a quadratic equation.

$3x^2 + 10x + 5 = 0$, $2x^2 - 5x + 7 = 0$ are some examples of quadratic equations.

If $p(a) = 0$ in $p(x)$, then a is a root of $p(x)$

For example,

in $P(x) = x^2 - 3x + 2$,

$$\begin{aligned} P(1) &= 1^2 - 3(1) + 2 \\ &= 1 - 3 + 2 \\ &= 0 \\ P(2) &= 2^2 - 3(2) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \\ P(3) &= 3^2 - 3(3) + 2 \\ &= 9 - 9 + 2 \\ &= 2 \end{aligned}$$

1 and 2 are the roots of $x^2 - 3x + 2 = 0$

3 is not the root of $x^2 - 3x + 2 = 0$

A quadratic equation has two roots.

Historical note :

4000 years ago, the Babylonians had known about the quadratic equations. In 365 B.C. Euclid, the Mathematical genius used geometrical methods to find the solution set for quadratic equations. Brahmagupta who lived in 598 A.D. also knew the method of solving the quadratic equations.

3-33. Quadratic equations: Factorization method

Example 1

$$x^2 - 25 = 0$$

Solution :

$x^2 - 25 = 0$

i.e. $(x + 5)(x - 5) = 0$

i.e. $x + 5 = 0$ or $x - 5 = 0$

$x = -5$ or $x = 5$

Solution set is $\{-5, 5\}$

Example 2 :

Solve : $x^2 + 7x + 12 = 0$

Solution :

$x + 7x + 12 = 0$

i.e. $(x + 4)(x + 3) = 0$

Then $x + 4 = 0$ or $x + 3 = 0$

$x = -4$ or $x = -3$

Solution set is $\{-4, -3\}$

Example 3 :

Solve : $3x^2 + 10x + 3 = 0$

Solution :

$3x^2 + 10x + 3 = 0$

i.e. $3x(x + 3) + 1(x + 3) = 0$

$(3x + 1)(x + 3) = 0$

$3x + 1 = 0$ or $x + 3 = 0$

$x = -\frac{1}{3}$ or $x = -3$

Solution set $\left\{-3, -\frac{1}{3}\right\}$

Exercise 3-33

Find the solution set for the following quadratic equations $(x \in R)$. (1 — 15 mental sums)

(1) $(x - 3)(x - 2) = 0$

(2) $(x - 4)(x + 5) = 0$

(3) $(x - 5)(x - 7) = 0$

(4) $(x - 2)(x + 6) = 0$

(5) $(x + 9)(x - 11) = 0$

(6) $(x + 8)(x + 3) = 0$

(7) $(x + 5)(x + 12) = 0$

(8) $(x + 6)^2 = 0$

(9) $(x + 8)^2 = 0$

(10) $(x - 1)^2 = 0$

(11) $x(x - 3) = 0$

(12) $x(x - 7) = 0$

(13) $x(4x - 5) = 0$

(14) $(4x + 3)(2x - 7) = 0$

(15) $(5x - 1)(3x + 4) = 0$

(16) $x^2 - 36 = 0$

(17) $x^2 - 169 = 0$

(18) $2x^2 - 50 = 0$

(19) $2x^2 - 7x = 0$

(20) $3x^2 + 4x = 0$

(21) $x^2 + 14x + 48 = 0$

(22) $x^2 + 24x + 143 = 0$

(23) $x^2 + 11x + 30 = 0$

(24) $x^2 - 21x + 104 = 0$

(25) $x^2 + x - 72 = 0$

(26) $x^2 + 5x - 150 = 0$

(27) $x^2 - x - 12 = 0$

(28) $x^2 - 6x - 91 = 0$

(29) $2x^2 + 5x + 2 = 0$

(30) $3x^2 + 10x + 8 = 0$

(31) $2x^2 - 3x + 1 = 0$

(32) $4x^2 + 8x - 5 = 0$

(33) $x^2 + 10x + 25 = 0$

(34) $x^2 - 6x + 9 = 0$

(35) $4x^2 - 28x + 49 = 0$

ANSWERS

(16) $\{6, -6\}$

(18) $\{5, -5\}$

(21) $\{-6, -8\}$

(23) $\{-6, -5\}$

(27) $\{4, -3\}$

(30) $\left\{-2, \frac{-4}{3}\right\}$

(35) $\left\{\frac{7}{2}\right\}$

3-34. Quadratic equations : Applications

Example 1 :

The sum of a natural number and its square is 56 Find the number.

Solution :

Let that number be x

Its square is x^2

$$x^2 + x = 56$$

$$x^2 + x - 56 = 0$$

$(x + 8)(x - 7) = 0$

$x = -8$ or 7

$-8 \notin N$

$\therefore$ 7 is the suitable answer

The required number is 7

Verification : $7 + 7^2 = 7 + 49 = 56$

Example 2 :

The length of a room is 2m more than its breadth. The area of the floor of the room is $120 m^2$. Find the length and the breadth of the room.

Solution :

Let the breadth of the room be x m

Then its length $= (x + 2)$ m

Area of the floor $= x(x + 2)\ m^2$

$$x(x + 2) = 120$$

$$x^2 + 2x = 120$$

$$x^2 + 2x - 120 = 0$$

$$(x + 12)(x - 10) = 0$$

$x = -12$ or $x = 10.$

$x = -12$ is inadmissible

$\therefore$ $x = 10$ is the required solution

Breadth of the room $= 10$ m

Length of the room $= (10 + 2)$ m

$= 12$ m

Example 3 :

The hypotenuse of a right angled triangle is 13 cm. One of the other two sides is 7 cm more than the third side. Find the area of that triangle.

Solution :

Length of the hypotenuse is 13 cm

Let the other two sides be x cm and $(x + 7)$ cm

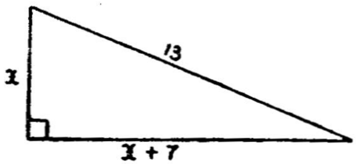


Fig. 3·2

According to Pythagoras Theorem,

$x^2 + (x + 7)^2 = 13^2$

$x^2 + x^2 + 14x + 49 = 169$

$2x^2 + 14x + 49 - 169 = 0$

$2x^2 + 14x - 120 = 0$

$x^2 + 7x - 60 = 0$ (Dividing the whole equation by 2)

$(x + 12)(x - 5) = 0$

$x = -12$ or $x = 5$

$x = -12$ is in admissible

$x = 5$ is the required solution

$\therefore$ The sides containing the right angle are

5 cm. and $(5 + 7)$ cm

$= 5$ cm. 12 cm.

Area of the triangle $= \frac{1}{2} \times 5 \times 12$

$= 30 \text{ cm}^2$

Exercise 3·34

(1) The product of two consecutive natural numbers is 420. Find the numbers.

(2) Divide 31 into two parts such that their product is 240.

(3) The sum of the squares of two consecutive odd numbers is 202. Find the numbers.

(4) The sum of a number and its reciprocal is $\frac{41}{20}$. Find the number.

(5) A number is 6 times the reciprocal of another number. The sum of these two numbers is 5. Find the numbers.

(6) The sum of the squares of three consecutive numbers is 77. Find the numbers.

(7) The product of two consecutive multiples of 7 is 294. Find the number.

(8) The difference of two natural numbers is 5. Their product is 500. Find the numbers.

(9) The perimeter of a rectangular field is 50 m. The area of the field is 150 m^2. Find the length and the breadth of the field.

(10) The side of a square field is 30 m. It is surrounded by a path of uniform width. If the area of the path is 256 m^2, find the width of the path.

(11) The length of a rectangular field is 20 m and its breadth is 15 m. There is a path of uniform width, running round inside the field. If the area of the path is 96 m^2 find the width of the path.

(12) The base of a right angled triangle is 1cm. longer than its height. Its area is 6 cm^2. Find the perimeter of the triangle.

(13) The slaut height of a cone is 17 cm. Its height is 7 cm more than its radius. Find the volume of the cone.

(14) The diagonal of a rectangle is 6·5 cm. Its length is 3·5 cm more than its breadth. Find the perimeter of the rectangle.

(15) A person purchased a few pens for Rs. 900. If the cost of a pen was Re 1 less, he would have bought 30 pens more. How many pens did he buy?

ANSWERS

(1)	20, 21	(2)	15, 16	(3)	9, 11	(4)	$\frac{4}{5}$ or $\frac{5}{4}$
(5)	3, 2	(6)	4, 5, 6	(7)	14, 21	(8)	20, 25
(9)	15m, 10m	(10)	2 m	(11)	$1\frac{1}{2}$ m	(12)	12 cm
(13)	320π cm^3	(14)	17 cm	(15)	150		

REVIEW

1. Identities :

(a) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(b) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(c) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(d) $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

(e) $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

(f) $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

2. To find the factors of $ax^2 + bx + c$,

(a) find the pair of factors of ac.

(b) (1) If ac is positive, then the sum of the two factors of ac must be b.

(2) If ac is negative, the difference between the two factors of ac must be b.

(3) Write bx as the sum of two terms and factorize.

3. If the polynomial $P(x)$, $x \in N$, is divided by $(x - a)$, the remainder is $P(a)$. **(Remainder Theorem)**

4. If $P(a) = 0$ $(a \neq 0)$ in a polynomial $P(x)$, then $(x - a)$ is a factor of $P(x)$. **(Factor Theorem)**

   a is called a "**zero**" of $P(x)$.

5. The rules and methods of finding the g.c.d., l.c.m. and factors of polynomials are the same as in Arithmetic.

6. If $P(x)$ and $Q(x) \neq 0$ are polynomials,

   $\frac{P(x)}{Q(x)}$ is a rational expression.

   The addition, subtraction, multiplication and division of rational expressions are done as in the case of rational numbers.

7. $ax^2 + bx + c = 0$ $(a, b, c \in R, a \neq 0)$ is a quadratic equation.

   If $(x^2 + px + q) = (x - l)(x - m)$, then the solution set of $x^2 + px + q = 0$ is $\{l, m\}$

SELF ASSESSMENT TEST

(1) Find the expansion of :

(a) $(2p - 3q - 4r)^2$

(b) $(7x + 3)^3$

(c) $\left(\frac{a}{b} - \frac{b}{a}\right)^3$

(d) $(x + 7)(x + 3)(x - 4)$

(2) If $x + \frac{1}{x} = 4$, find the value of

$$x^3 + x^2 + x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$$

(3) Find the factors of :

(a) $a^4 - 4b^4$

(b) $x^2 + x - 110$

(c) $30 - x - 42x^2$

(4) When $2x^3 - 3x^2 + px - 8$ is divided by $(x - 2)$, the remainder is 18. Find the value of p.

(5) Find the factors of :

$$x^3 + 4x^2 + x - 6$$

(6) Find the square root of :

$4x^4 - 20x^3 + 33x^2 - 20x + 4$

(7) Find the g.c.d. of $x^2 - 2x - 15$, $x^2 + 2x - 3$ and $x^2 + 10x + 21$.

(8) Simplify :

$$\frac{x^3 + 1}{x^2 - 3x + 2} \times \frac{x^2 + 2x - 8}{x^2 - 4x - 5} \div \frac{x^2 - 9}{x^2 - 25}$$

(9) Simplify :

$$\frac{1}{a + b} - \frac{2}{a - b} + \frac{3}{a^2 - b^2}$$

(10) The sum of a natural number and its square is 72. Find the number.

NOT FOR EXAMINATION

MATHEMATICS CLUB—ACTIVITY 3

Who is the Criminal ?

Three persons are called for enquiry in connection with the murder of Anandan. They give three statements each to the police.

Fox:

(1) I did not commit the murder.

(2) Anandan has not committed suicide.

(3) I am Anandan's friend.

Tiger : (1) Anandan has not committed suicide.

(2) I was talking to Anandan.

(3) Fox is not Anandan's friend.

Cat : (1) Tiger did not murder Anandan.

(2) Anandan has committed suicide.

(3) The statement of Fox that he is Anandan's friend is true.

The police officer found out that two of the three statements given by each of the three persons were true and each one has given only one false statement. Find out the real culprit.

NOT FOR EXAMINATION

Sivasankara Narayanan (1901—1950)

Sivasankara Narayanan, known as S. S. Pillai in the world of Mathematics was a great researcher and scholar. He belonged to South India. He was a great thinker, who found the solution for the problem posed by Waring.

In 1640 the French Mathematician Fermat wrote that any integer can be expressed as the sum of 4 squares. Lagrange gave a mathematical proof for this in 1770. In the same year, Waring, Professor of Mathematics in Cambridge University proposed the following extension of the result.

We can express any positive integer as

(a) The sum of 4 squares

(b) The sum of 9 cubes

(c) The sum of 19 fourth powers

Many tried to solve this conjecture but failed. Sivasankara Narayanan proved it partially in 1935 and fully in 1940. He found the relevant function which would give the number

of terms needed to express any positive integer as a sum of powers. The function is

$$g(k) = 2^k + m - 2, \quad m = \left[\left(\frac{3}{2}\right)^k\right]$$

$[x]$ is the integral part of x.

If $k = 1$, $\left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)$, $m = \left[\left(\frac{3}{2}\right)\right] = 1$

$g(1) = 2 + 1 - 2 = 1$ This is true.

If $k = 2$, $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$; $m = \left[\frac{9}{4}\right] = 2$

$g(2) = 2^2 + 2 - 2 = 4$ This is Fermat's conjecture.

If $k = 3$, $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$; $m = \left[\frac{27}{8}\right] = 3$

$g(3) = 2^2 + 3 - 2 = 9$ This is Waring's problem.

If $k = 4$, $\left(\frac{3}{2}\right)^4 = \frac{81}{16}$, $m = \left[\frac{81}{16}\right] = 5$

$g(4) = 2^4 + 5 - 2 = 19$. This is also Waring's problem.

On account of the solution given by S S. Pillai of our country, we can express an integer as the sum of powers even higher than 4.

Example:

Any positive integer can be expressed as the sum of 73 numbers, each being a sixth power.

$$k = 6;\ \left(\frac{3}{2}\right)^6 = \frac{729}{64} = 11\frac{25}{64};\ m = 11$$

$$g(6) = 2^6 + 11 - 2 = 73.$$

In his Twelth Ramanujan Commemorative lecture, Prof. Hardy referred to this as "the most interesting theorem of Pillai".

Sivasankaranarayanan also worked on an unsolved problem of Fourier Series. Before he could complete it, he died unfortunately in an air crash leaving the Indian Mathematics poorer by his untimely death.

You have so far studied the lives of a few great Indian mathematicians such as Brahmagupta, Aryabhatta, Bhaskara, Ramanujan and Sivasankaranarayanan. Our country can truly be proud of such a galaxy of scholars. Try to emulate them in your own life.

4. STATISTICS

4·1. Frequency Table

Let us recall what we have studied about 'Frequency Table' in VIII Std.

Example :

The heights (in cm.) of 30 students in a standard are given below. Frame a frequency table taking suitable class intervals.

163	166	168	162	163	169
161	160	157	166	169	158
165	168	169	168	166	169
160	169	164	155	168	167
158	162	167	159	165	156

Solution :

Highest value = 169 (maximum height)

Lowest value = 155 (minimum height)

Range = 169 − 155 = 14

Total number of data given = 30

To have atleast 5 equal class intervals,

$$\text{the measure of the class interval} = \frac{\text{Range}}{\text{Number of classes}}$$

$$= \frac{14}{5} = 2{\cdot}8$$

So the class interval to be chosen is 3.

Class intervals	Tally marks	Frequency (f)
154·5 — 157·5	|||	3
157·5 — 160·5	卌	5
160·5 — 163·5	卌	5
163·5 — 166 5	卌 |	6
166·5 — 169 5	卌 卌 |	11
		$\Sigma f = 30$

Note :

If we take the class intervals as 155 - 157, 157 - 160,...... then a confusion may arise with regard to the inclusion of the 'limits' of the class intervals because those numbers will be found in two consecutive class intervals. So we take the class intervals as 154·5 - 157·5, 157·5 - 160·5,...

Exercise 4·1

Form a frequency table for the following data taking suitable class intervals.

1. Weight of 35 pupils (correct to kg)

45	32	33	36	32	38	31
45	47	32	35	43	33	40
43	45	49	34	35	38	30
33	41	41	37	36	37	39
31	34	39	49	42	40	44

2. The following are the average ages of 25 workers in Madras port (Age in years.)

34	41	39	30	29
39	36	37	38	40
36	44	46	35	42
40	38	35	37	45
33	28	39	32	31

3. Areas (correct to hectares) of paddy cultivation in 40 villages are given below.

14	13	14	18	15	20	22	23
16	15	20	19	15	17	13	23
16	24	17	20	11	18	24	14
14	12	21	17	12	19	11	24
22	23	22	16	21	23	21	13

4, The diameters of ball-bearings manufactured in a factory are given below (correct to mm)

7·32	7·35	7·30	7·32	7·27
7·37	7·29	7·38	7·40	7·44
7·31	7·34	7·44	7·39	7·46
7 46	7·33	7·41	7 30	7·24
7·35	7·30	7·39	7·34	7·35

4·2. Measures of Central Tendency

4·2-1. Arithmetic Mean of Discrete Variables (A. M.)

If we divide the sum of the data by the number of data, we get the Arithmetic mean.

$$\bar{x} = \frac{\Sigma x}{n}$$

Example 1 :

The expenditure on education of 5 boys in a family is Rs. 60, Rs. 70, Rs. 90, Rs. 120 and Rs. 150 resvectively. Find their average expenditure

Solution :

$$\text{Arithmetic mean} = \text{Rs.}\ \frac{60 + 70 + 90 + 120 + 150}{5}$$

$$= \text{Rs.}\ \frac{490}{5} = \text{Rs. } 98.$$

Example 2 :

In a class there are 4 pupils 66 cm high, 5 pupils 67 cm high and 6 pupils 68 cm high. What is their average height?

Solution :

(i) Average height of the pupils is

$$\bar{x} = \frac{(66\times4)+(67\times5)+(68\times6)}{15}$$

$$= \frac{1007}{15} = 67{\cdot}13$$

$$= 67{\cdot}1 \text{ cm}$$

(ii) **Tabular method :**

We can form the frequency table from the data and find the Arithmetic mean (AM)

Height in cm (x)	Numbers of pupils (f)	fx
66	4	264
67	5	335
68	6	408
	$n = \Sigma f = 15$	$\Sigma fx = 1007$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

$$\bar{x} = \frac{1007}{15} = 67{\cdot}13 = 67{\cdot}1 \text{ cm}$$

(iii) Assumed Mean Method

We can take an assumed mean of the data and find the differences between the variable and the assumed mean and then find the A.M.

Let the assumed mean A be 67.

Height in cm x	Difference between the number and the approximate average $d = x - A$	Number of pupils f	fd
66	-1	4	-4
67	0	5	0
68	1	6	6
		$\Sigma f = 15$	$\Sigma fd = 2$

$$\bar{x} = A + \frac{\Sigma fd}{\Sigma f}$$

$$\bar{x} = 67 + \frac{2}{15} = 67{\cdot}13 = 67{\cdot}1 \text{ cm (correct to mm)}$$

Exercise 4-2·1

1. Find the A. M. of the following :

(i) The cubes of the first n natural numbers.

(ii) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(iii) The maximum temperature (°C) of a place for the first10days in a month :

32·3, 33·8, 28·8, 33·8, 30·4

30·3, 31·5, 33·4, 32·4, 32·1

(iv) The weight (kg) of 20 babies born in a month in a Government maternity centre.

2·3, 3·4, 2·4, 2·1, 3·0, 2·2, 3·5,

2·9, 3·0, 4·0, 2·5, 3·0, 4·1, 3·8,

2·8, 2·0, 3·0, 3·3, 2·1, 3·0

2. Find the A. M. for the data in the following frequency table.

(i) The marks scored by 50 pupils of a class in an examination.

Marks	10	20	30	40	50	60	70
No. of pupils	3	9	8	15	9	4	2

(ii) Weight of 30 employees in a factory.

Weight in kg	52	58	60	65	68	70	75
Frequency	7	5	4	6	3	3	2

3. The mean profit in a day in a bazaar is as follows : Rs. 7 for the first 10 shops and Rs. 12 for the next 15 shops. What is the combined mean profit of all the shops ?

4. (i) The A. M. of 8 numbers is -8. If 3 is added to every number, find the new A.M.

(ii) The A. M. of 12 numbers is 6. If every number is multiplied by 3, what is the new A. M. ?

ANSWERS

1. (i) $\frac{n\,(n+1)^2}{4}$ (ii) 14·3 (iii) 31·88°C (iv) 2·92 kg

2. (i) 37·6 (ii) 61·6 kg

3. Rs. 10 4. (i) -5 (ii) 18

4-2-2. Arithmetic Mean of Continuous Variable

In a frequency table, every value in a class is considered to be equal to the midpoint of the class. The mid point of the class is obtained by dividing the sum of the lowest and the highest values of the class (i.e. limits) by 2. Making use of this, we will find the A.M. of continuous variables of a frequency table.

Example :

What is the mean height of the students according to the data given in the table in 4-1.

Solution :

(i)

Height (in cm)	Number of students (f)	Mid point (x)	fx
154·5 — 157·5	3	156	468
157·5 — 160·5	5	159	795
160·5 — 163·5	5	162	810
163·5 — 166·5	6	165	990
166·5 — 169·5	11	168	1848
	$\Sigma f = 30$		$\Sigma fx = 4911$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{4911}{30} = 163 \cdot 7 \text{ cm}$$

(ii) Assumed Mean method

Let the assumed mean A be 162.

Height (in cm)	Midpoint	Number of students (f)	Deviation of midpoint from the variable (d)	fd
154·5—157·5	156	3	−6	− 18
157·5—160·5	159	5	−3	− 15
160·5—163·5	162	5	0	0
163·5—166·5	165	6	3	18
166·5—169·5	168	11	6	66
		$\Sigma f = 30$		$\Sigma fd = 51$

$$\bar{x} = A + \frac{\Sigma fd}{\Sigma f} = 162 + \frac{51}{30} = 162 + 1 \cdot 7$$

The mean height of the class = 163·7 cm

Weighted Arithmetic Mean

We know that in some competitions, the point system is adopted. 'Weights' are decided according to the importance or necessity of each competition. We can find the Weighted Arithmetic Mean accordingly. The Weighted Arithmetic Mean is useful in our day-to-day life in finding out the cost of living index, etc.

Example 1 :

In a competitive examination, a student scored 72, 90 and 86 marks in English, Mathematics. and Science respectively. These examinations are awarded, 1, 1 and 3 points respectively. What is the weighted Arithmetic Mean of the marks scored by the student ?

Solution :

Weighted Arithmetic Mean

$$= \frac{(72\times 1) + (90\times 1) + (86\times 3)}{1 + 1 + 3}$$

$$= \frac{72 + 90 + 258}{5} = \frac{420}{5} = 84$$

Example 2 :

In a factory 50 persons are paid Rs. 3 per hour, 30 are paid at the rate of Rs. 4 per hour, and 20 are paid Rs. 5 per hour. What is their mean earnings per hour?

Number of employees (x)	Earnings per hour in Rs. (w)	wx
50	3	150
30	4	120
20	5	100
$\Sigma x = 100$		$\Sigma wx = 370$

$$\bar{x} = \frac{\Sigma wx}{\Sigma x} = \frac{370}{100} = 3{\cdot}70$$

Exercise 4-2-2

1. In the following frequency table, find the A.M.

(i)

Monthly salary (Rs.)	Number of employees (f)
125 — 175	2
175 — 225	22
225 — 275	19
275 — 325	14
325 — 375	3
	$\Sigma f = 60$

(ii)

Monthly savings (Rs.)	Number of persons saving (f)
50 — 52	7
52 — 54	10
54 — 56	15
56 — 58	12
58 — 60	1
	$\Sigma f = 45$

2. Find the Weighted Arithmetic Mean for the following data:

(i)

Group	Monthly increment in salary Rs.	Number of persons getting increment
A	3	500
B	8	1500
C	10	2000

(ii)

Material	Increase in price%	Weight
A	10	6
B	24	60
C	30	50
D	40	75

ANSWERS

(1) (i) Rs. 245 (ii) Rs. 54·56

(2) (i) Rs. 8·38 (ii) 31·4%.

4-2-3. Median

What helps to find out the average immediately on seeing the data is the **Median.**

> Write the data in an ascending order or a descending order. The central value or the mean of two central values is called the **Median.**

So, to find the **Median,**

(1) Write the data in an ascending or descending order.

(2) If the number of data is odd, the central value is the Median; if it is even, the mean of the two central values is the median.

Example 1 :

In a competition for the scouts, one boy scored 5, 2, 4, 3, 4, 1 and 2 points. What is his median score ?

Solution :

Write the data in the ascending order.

1, 2, 2, 3, 4, 4, 5. The central value is the 4th term, 3.

∴ The median is 3.

Example 2 :

In a research work on plantain cultivation in a farm, the lengths of plantain leaves (in cm.) are found as follows. Find the median.

173,	128,	145,	150,	132,	156,	153,	140,
138,	135,	135,	144,	125,	157,	176,	146,
158,	163,	135,	152,	149,	148,	119,	168,
161,	142,	154,	165,	142,	147,	138,	126,
150,	140,	136,	147,	144,	145,	164,	146.

Solution :

Writing the data in the ascending order we get

119	135	138	144	146	150	156	164
125	135	140	144	147	150	157	165
126	135	140	145	147	152	158	168
128	136	142	145	148	153	161	173
132	138	142	146	149	154	163	176

The Central values are the 20th and the 21st terms.

i.e. 146 and 146

$\therefore$ The median is $\frac{146 + 146}{2} = 146$ cm.

Exercise 4-2-3

Find the median of the following data.

1. (i) The sequence of the first 100 natural numbers.
   (ii) The sequence of the first 100 whole numbes.
   (iii) 6 sixes, 7 sevens, 8 eights, 9 nines and 10 tens.

2. The profit of a factory in 8 months (%) is

   19·3, 20·2, 18·3, 22·4, 20·6, 20·0, 18·8, 19·7

3. In a medical camp, the number of persons who obtained family planning advice, in the first 20 days is as follows : 77, 75, 78, 83, 85, 81, 86, 78, 76, 81, 85, 74, 68, 79, 77, 70, 71, 84, 80, 69.

4. Number of those who died in road accidents in a city during the first 9 months are as follows :

8, 8, 4, 5, 4, 6, 3, 8, 10

ANSWERS

1. (i) 50·5	(ii) 50	(iii) 8
2. 19·85%	3. 78	4. 6

4·2·4 Mode

In a given set of data, the value which occurs with the greatest frequency is taken as the mean of the whole series. This is called the mode.

The mode is said to be the value which occurs most often, that is, with the highest frequency.

The traders who sell shirts, boots, etc., must know what colour and what size are in demand and liked by most people. Mode is very useful for them.

To find the mode

1. Note the frequency of the values.

2. If there is the highest frequency for any value, that value is the mode.

Example :

Find the mode of the following :

(i) 3, 3, 6, 8, 9, 9, 9, 11, 11, 12, 13, 9.

(ii) 4, 6, 9, 11, 13, 16, 17.

(iii) 3, 4, 5, 5, 5, 6, 6, 8, 8, 8, 10.

Solution :

(i) In this group, the frequency of 9 is 4

Frequency of 3 is 2.

Frequency of 11 is 2.

The mode of this group is 9.

(ii) The frequency of every number is 1 in this series. So there is no mode for this series.

(iii) In this series, the frequeney of 5 is 3 and that of 8 is 3.

So there are two modes 5 and 8 for this series.

Example 2 :

Find (i) the arithmetic mean, (ii) the median and (iii) the mode of the set of five numbers.

58·7, 60·3, 59·5, 61·6, 58·9.

Solution :

(i) Arithmetic mean

$$= \frac{(58{\cdot}7 + 60{\cdot}3 + 59{\cdot}5 + 61{\cdot}6 + 58{\cdot}9)}{5}$$

$$= \frac{1}{5} \times 299{\cdot}0$$

$$= 59{\cdot}8.$$

(ii) Ascending order of the numbers:

58·7, 58·9, 59·5, 60·3, 61·6

The central value is the 3rd term, 59·5. So the median is 59·5.

(iii) Every value in this set has the same frequency. Therefore, there is no mode.

Exercise 4-2-4

Find (i) Arithmetic mean (ii) Median and (iii) Mode of the following.

1. 18, 21, 14, 13, 12, 15, 20, 16, 14, 11, 20, 18, 22, 16, 1[illegible], [illegible].

2. 6 sixes, 7 sevens, 8 eights, 9 nines and 10 tens.

3. 0, 11, 16, 17, 12, 13, 17, 16, 16, 12, 16, 0, 15, 16, 0.

ANSWERS

1. (i) 16·4 (ii) 16·4 (iii) No mode.
2. (i) 8·25 (ii) 8 (iii) 10
3. (i) 11·08 (ii) 15 (iii) 16

4-3-1. Measures of Dispersion : Range

Let us compare the runs scored by 2 cricketers, *A* and *B* in 5 innings.

	Run in innings					Total runs	Arithmetic mean
	1	2	3	4	5		
A	15	30	30	0	75	150	30
B	20	30	30	30	40	150	30

The maximum of the variations of the scores of *A* from the mean is 45 and the minimum of variations is 0. But the maximum of variations of *B* is 10 and the minimum of variations is 10. Therefore, we can say that *B* is a more consistent player than *A*.

Further,

The range of runs of $A = 75 - 0 = 75$

The range of runs of $B = 40 - 20 = 20$.

Thus the measures which help to find how the values vary from some central value, are called **measures of dispersion.** One of the measures of dispersion is the **Range.** The difference between the highest and the lowest values of the given data is called the range of the dispersion.

Exercise 4-3-1

Find the range in any 10 problems of the exercises 4·1, 4·2, 4·3 and 4·4

4-3-2. Standard Deviation : Discrete Variables

Take the sequence 3, 5, 7, 9, 11 and the sequence

3, 4, 6, 8, 10, 11.

In both the sequences the range is 11 — 3 = 8.

Yet the difference between the boundary values in the two sequences 3 and 11 does not reveal anything about the other values. But the standard deviation is useful to understand the dispersion on the basis of all the values given.

To find the standard deviation :

1. Find the Arithmetic mean, $\bar{x}$.

2. Find the deviation d between the A.M. and every value $d = x - \bar{x}$.

3. Find d^2.

4. Find the mean of the squares of deviation, σ^2 i.e. $\frac{\Sigma d^2}{n}$

5. Find standard deviation $\sigma = \sqrt{\frac{\Sigma d^2}{n}}$

Example :

The marks scored by a student in the first four monthly tests in Mathematics are 42, 30, 48 and 36. Find the standard deviation.

Solution :

The number of values, $n = 4$

The arithmetic mean $\bar{x} = \frac{42 + 30 + 48 + 36}{4}$

$= \frac{156}{4} = 39$

Marks x	Deviation from the A.M. $d=x-\bar{x}$	Square of the deviation d^2
42	+ 3	9
30	− 9	81
48	+ 9	81
36	− 3	9
	$\Sigma d = 0$	$\Sigma d^2 = 180$

Note : Σd of the discrete values = 0.

$$= \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{180}{4}} = \frac{6\sqrt{5}}{2} = 3\sqrt{5} = 3 \times 2{\cdot}236$$

$$= 6{\cdot}708.$$

Exercise 4-3-2

Find the standard deviation of the following :

1. In an Art Exhibition of Modern painting, the bases of ten similar triangle - shaped figures are (in cm) :

   4, 6, 8, 10, 12, 14, 16, 18, 20, 22

2. The rainfall (in cm) in October for eight years in a city :

   9, 18, 9, 3, 8, 8, 9, 8

3. The number of hens attacked by diseases in a year in 10 poultry farms (%) :

   8·0, 8·8, 8·2, 9·0, 6·8, 8·9, 7·4, 8·2, 6·6, 7·6

4. The number of experts employed in a Textile Mill during 10 years for repairing the machines :

   7, 20, 21, 13, 15, 10, 12, 29, 35, 28

ANSWERS

(1) 5·745 (2) 3·87 (3) 0·802 (4) 8·76

4-3-3 Standard Deviation : Frequency Table

If n values are given in a frequency table,

the standard deviation $\sigma = \sqrt{\dfrac{\Sigma fd^2}{\Sigma f}}$

Example :

The frequency table with regard to the cultivation of a new variety of paddy in a Panchayat Union is given below. Find the standard deviation of the values.

Area (Hectares)	5	7	9	11	13	
Number of cultivators	2	2	4	8	4	$\Sigma f = 20$

Solution :

x	f	fx	$d = x - \bar{x}$	d^2	fd^2
5	2	10	-5	25	50
7	2	14	-3	9	18
9	4	36	-1	1	4
11	8	88	$+1$	1	8
13	4	52	$+3$	9	36
	$\Sigma f = 20$	$\Sigma fx = 200$			$\Sigma fd^2 = 116$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{200}{20} = 10$$

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f}} = \sqrt{\frac{116}{20}} = \sqrt{\frac{58}{10}}$$

$$= 2{\cdot}408.$$

Exercise 4-3-3

Find the standard deviation of the data given in the frequency tables below.

1.

x	4	12	20	28	36	
f	2	3	5	4	1	$\Sigma f = 15$

2.

x	25	35	45	55	65	75	85	
f	3	8	5	18	3	2	1	$\Sigma f = 40$

3.

x	125	175	225	275	325	375	
f	7	18	25	30	12	8	$\Sigma f = 100$

4.

x	61	64	67	70	73	
f	5	18	42	27	8	$\Sigma f = 100$

ANSWERS

(1) 8·98 (2) 13·6 (3) 66·10 (4) 2·92

4-4-1. Probability : Revision I

Let us recall what we studied about probability in the Ninth Standard.

1. The set of all possible outcomes of an experiment is called a Sample Space (S).

2. The subset of Sample Space is called Event (E).

3. The Probability of an Event E is $P(E) = \frac{n(E)}{n(S)}$

4. $0 \leq P(E) \leq 1$.

Example 1 :

What is the probability of getting a head in tossing a coin?

Solution :

Sample Space $S = \{H, T\}$; $n(S) = 2$

Probability $E = \{H\}$; $n(E) = 1$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$$

Example 2 :

In rolling a die, what is the probability of getting (i) a prime number, (ii) number greater than 4, (iii) a multiple of 3, (iv) zero ?

Solution :

Sample Space $S = \{1, 2, 3, 4, 5, 6\}$; $n(S) = 6$

(i) Event of getting Primes

$$E_1 = \{2, 3, 5\}; \; n(E_1) = 3$$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) Event of getting a number > 4,

$$E_2 = \{5. 6\}; \; n(E_2) = 2$$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(iii) The Event of getting a multiple of 3,

$$E_3 = \{3, 6\}; \quad n(E_3) = 2$$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(iv) The Event of getting 0,

$$E_4 = \{\ \}; \quad n(E_4) = 0$$

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{0}{6} = 0$$

Example 3 :

If a ball is taken at random from a bag containing 7 blue balls and 5 red balls, what is the probability of getting (i) a blue ball and (ii) a red ball ?

Solution :

(i) The total number of balls in the bag

$$n(S) = 7 + 5 = 12$$

Let the event of taking a blue ball be E_1

$$n(E_1) = 7$$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{7}{12}$$

(ii) Let the event of taking a red ball be E_2

$$n(E_2) = 5$$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{5}{12}$$

Example 4 :

There are 20 wrist watches in a box. Out of them 6 are defective. If a watch is taken at random, what is the probability of getting a good watch ?

Solution :

Total number of wrist watches $n(S) = 20$

Number of good watches $= 20 - 6 = 14$

Let the Event of getting a good watch be E

$$n(E) = 14$$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{14}{20} = \frac{7}{10}$$

Exercise 4-4·1

1. The numbers from 1 to 10 have been written on pieces of paper one on each. If a paper is taken at random, what is the probability of the number being (i) odd, (ii) even, (iii) divisble by 3 and (iv) a multiple of 5?

2. The scores in a game are given on the disc. If they have equal chances, what is the probability of striking an arrow at (i) 2, (ii) 3, (iii) 5 and (iv) 10?

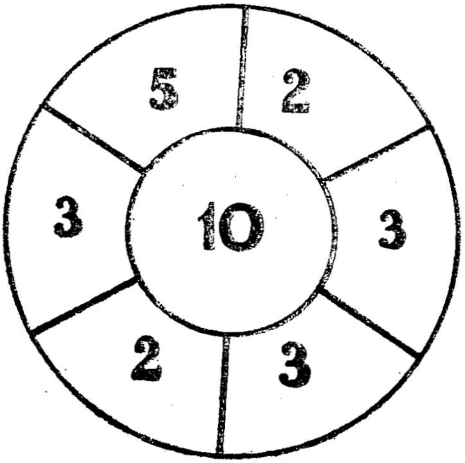


Fig. 4-1.

3. When a die is thrown once what is the probability of getting (i) an odd number, (ii) an even number, (iii) a number < 5 and (iv) a number > 5?

4. If a marble is taken at random from a bag containing 8 red marbles, 7 blue ones and 5 green ones, what is the probability of getting (i) a red marble, (ii) a blue one, (iii) a green one and (iv) a marble which is not blue?

ANSWERS

(1)	(i) $\frac{1}{2}$	(ii) $\frac{1}{2}$	(iii) $\frac{3}{10}$	(iv) $\frac{1}{5}$
(2)	(i) $\frac{2}{7}$	(ii) $\frac{3}{7}$	(iii) $\frac{1}{7}$	(iv) $\frac{1}{7}$
(3)	(i) $\frac{1}{2}$	(ii) $\frac{1}{2}$	(iii) $\frac{2}{3}$	(iv) $\frac{1}{6}$
(4)	(i) $\frac{2}{5}$	(ii) $\frac{7}{20}$	(iii) $\frac{1}{4}$	(iv) $\frac{13}{20}$

4-4-2. Probability : Revision II

Example 1.:

If two coins are tossed at a time, (i) What are the events ? (ii) What is the probability of getting two heads ? (iii) What is the probability of getting atleast one head ? (iv) What is the probability of getting only one head ?

Solution :

(i) When two coins are tossed at a time, the events that happen :

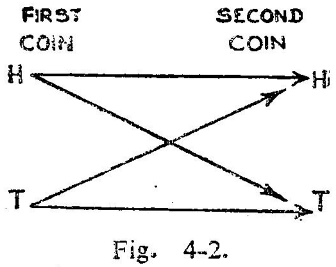


Fig. 4-2.

The Sample Space $S = \{ HH, HT, TH, TT \}$

(ii) $n(S) = 4$

The Event of getting two heads $E_1 = \{ HH \}$

$$\therefore \quad P(E_1) = \frac{1}{4}$$

(iii) Let the Event of getting at least one head be E_2

$$E_2 = \{ HH, HT, TH \}$$

$$n(E_2) = 3$$

$$\therefore \quad P(E_2) = \frac{3}{4}$$

(iv) Let the Event of getting only one head be E_3

$$E_3 = \{ HT, TH \}$$

$$n(E_3) = 2$$

$$\therefore \quad P(E_3) = \frac{2}{4} = \frac{1}{2}$$

Example 2 :

3 dice are thrown at a time. What is the probability of getting the numbers the sum of which is (i) 10 and (ii) 13?

Solution :

(i) If 2 dice are thrown at a time, the number of events are $6 \times 6 = 36$.

The ordered pairs constituting the sample space,

$$S = \{ (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1),$$
$$(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2),$$
$$(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3),$$
$$(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4),$$
$$(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5),$$
$$(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6) \}$$

$$\therefore \quad n(S) = 36.$$

Let the event in which the sum of numbers is 10 be E_1

$$E_1 = \{(4, 6), (5, 5), (6, 4) \}$$

$$n(E_1) = 3$$

$$P(E_1) = \frac{3}{36} = \frac{1}{12}$$

(ii) Let the event in which the sum is 13 be E_2

$$E_2 = \{ \quad \}$$

$$n(E_2) = 0$$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{0}{36} = 0$$

Exercise 4-4-2

1. When 2 coins are tossed at a time what is the probability of getting (i) two tails (ii) one tail (iii) only one tail (iv) not getting a tail at all ?

2. 3 coins are tossed at a time. (i) What is the sample space ? (ii) What is the probability of getting 3 heads, (ii) getting 3 tails, (iv) not to get a head at all and (v) not to get a tail at all ?

3. 2 dice are thrown at a time. What is the probability of their sum being (i) 8, (ii) less than 8 and (iii) greater than 8 ?

4. When 2 dice are thrown at a time, what is the probability of the following ?

   (i) Getting 6 in the first die and even number on the second die.

   (ii) Getting a number greater than 4 on each of the dice.

ANSWERS

(1) (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{4}$

(2) (i) { *HHH, HHT, HTH, HTT, THH, THT, TTH, TTT* }

(ii) $\frac{1}{8}$ (iii) $\frac{1}{8}$ (iv) $\frac{1}{8}$ (v) $\frac{1}{8}$

(3) (i) $\frac{5}{36}$ (ii) $\frac{7}{12}$ (iii) $\frac{5}{18}$

(4) (i) $\frac{1}{12}$ (ii) $\frac{1}{9}$

4.4.3. Probability: The addition Theorem: Mutually Exclusive Events

In the experiment of tossing a coin, if the head appears, we can certainly say that it is not the tail. If it is tail, we can say that it is not the head. In die-throwing, if an odd number appears, we can say that it is not an even number. This is because there is no chance for one if the other appears. These events are called **mutually exclusive events.**

On the other hand, in die-throwing, if we say that an even number appeared we cannot say that a prime number did not appear because 2 is an even number as well as a prime. The event is common for both. So these events are not mutually exclusive.

We know already that in set language, if A' is the complement of A,

$$A \cup A' = E; \quad A \cap A' = \phi$$

In probability the Universal set is S, the sample space. If A is an event, then A' is its mutually exclusive event; A' is the complementary event of A.

So $A \cup A' = S$; $A \cap A' = \phi$

Further, $n(A \cup A') = n(A) + n(A') = n(S)$.

$$P(A \cup A') = \frac{n(A \cup A')}{n(S)} = \frac{n(S)}{n(S)} = 1$$

and $$P(A \cup A') = \frac{n(A) + n(A')}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(A')}{n(S)}$$

$$= P(A) + P(A')$$

$$P(A \cap A') = \frac{n(A \cap A')}{n(S)} = \frac{0}{n(S)} = 0$$

> If A and A' are mutually exclusive events,
> $P(A \cup A') = P(A) + P(A') = 1$
> $P(A \cap A') = 0$

From this we get $P(A') = 1 - P(A)$

If $\sim A$ denotes A', then $P(\sim A) = 1 - P(A)$.

Example 1 :

In throwing a die what is the probability of getting (i) the number 3 and (ii) a number other than 3 ?

Solution :

(i) Let the event of getting the numbers 3 be E_1

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{6}$$

(ii) Let the event of getting a number other than 3 be E_2

$$P(E_1 \cap E_2) = P(\{\ \}) = 0$$

$$P(E_2) = 1 - P(E_1)$$

$$= 1 - \frac{1}{6}$$

$$= \frac{5}{6}$$

Example 2 :

If 2 dice are thrown at a time, what is the probability that the sum of the two numbers is 7 or 9 ?

Solution :

Let the event of getting 7 as the sum of the two numbers be E_1.

$$E_1 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$P(E)_1 = \frac{n(E_1)}{n(S)} = \frac{6}{36}$$

Let the event that the sum of the two numbers is 9 be E_2.

$$E_2 = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4}{36}$$

$$P(E_1 \cap E_2) = P(\{\ \}) = 0.$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$= \frac{6}{36} + \frac{4}{36} = \frac{10}{36} = \frac{5}{18}$$

Exercise 4-4-3.

1. In a factory there are 5 engineers, 10 supervisors and 85 workers. What is the probability that (i) an engineer, (ii) a supervisor and (iii) a worker is elected the president of the recreation club of the factory ?

2. A ball is taken at random from a bag containing 8 red balls, 6 white ones and 4 blue ones. What is the probability that (i) it is a red or white ball and (ii) it is not either red or blue ?

3. In a car-race the probability that a car comes first is $\frac{1}{10}$ the probability that it comes seond is $\frac{1}{8}$. What ts the probability that it does not come first or second ?

4. When 2 dice are thrown at a time, what is the probability that we get an odd number in each or the sum of the two numbers is 9 ?

ANSWERS

(1) $\frac{3}{20}$ (2) (a) $\frac{7}{9}$ (b) $\frac{1}{3}$ (3) $\frac{31}{40}$ (4) $\frac{13}{36}$

4-4·4. Probability : Addition Theorem : Not Mutually Exclusive Events

We know that if A and B are two sets,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Now let us try to find the probability that one or both of the events A and B occur at a time.

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B) - n(A \cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$= P(A) + P(B) - P(A \cap B)$$

Therefore,

> If A and B are not mutually exclusive events,
> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 1 :

In a throw of a die, what is the probability that we get an odd number or a multiple of 3 ?

Solution :

First event is getting an odd number

$$A = \{1, 3, 5\} \quad P(A) = \frac{3}{6}$$

2nd event is getting a multiple of 3

$$B = \{3, 6\} \quad P(B) = \frac{2}{6}$$

The set of events in which both of them occur

$$A \cap B = \{3\}; \quad P(A \cap B) = \frac{1}{6}$$

The event in which either of them occurs

$$A \cup B = \{1, 3, 5, 6\}$$

$$\therefore \quad P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$

Verification :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

Example 2 :

A and B are two events and $P(A) = 0{\cdot}25$; $P(A \cup B) = 0{\cdot}72$; $P(A \cap B) = 0{\cdot}14$; Find $P(\sim B)$.

Solution :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(B) = P(A \cup B) - P(A) + P(A \cap B)$$
$$= 0{\cdot}72 - 0{\cdot}25 + 0{\cdot}14$$
$$= 0{\cdot}61$$

$$P(B) + P(\sim B) = 1$$
$$P(\sim B) = 1 - 0{\cdot}61$$
$$= 0{\cdot}39.$$

Exercise 4-4·4

1. When 2 dice are thrown at a time, what is the probability that it is an even number on the first die or the sum of the two numbers of the dice is 10 ?

2. A and B are two events. $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}5$, $P(A \cap B) = 0{\cdot}1$. Find the values of the following.
   (i) $P(\sim A)$ (ii) $P(\sim B)$ (iii) $P(A \cup B)$

3. A coin and a die are thrown at a time. What is the probability that the event is a head or a 6 ?

4. The numbers from 1 to 12 are written on a piece of paper each. If a number is chosen at random, what is the probability that it is neither an even number nor divisible by 3 ?

ANSWERS

(1) $\frac{19}{36}$ (2) (i) 0·6 (ii) 0·5 (iii) 0·8 (3) $\frac{7}{12}$ (4) $\frac{1}{3}$

4-4·5. Probability : Multiplication Theorem : Independent Events

Let us assume that when a coin is tossed we get the head. If we toss the coin a second time, what is the event ? It may be a head or a tail. The first event does not influence the happening of the second. Getting a head or a tail in the second throw is not dependent on the first event. Such events are said to be independent.

In general, if A and B are two events and the probability of happening or not happening of the one does not affect the probability of another event happening or not happening, then these two events are called independent events.

The probability of two independent events at a time is the product of their individual probabilities

So,

If A and B are independent events $P(A \cap B) = P(A) \cdot P(B)$

This is called the Multiplication Theorem of probability. The following examples explain how this rule is applied.

Example 1 :

If 2 dice are thrown at a time what is the probability that we get (i) two 6s, (ii) one 6 only and (iii) no six at all ?

Solution :

We can work out the problem after writing down all the 36 ordered pairs when two dice are cast at a time. But finding the probability through a tree-diagram is easier.

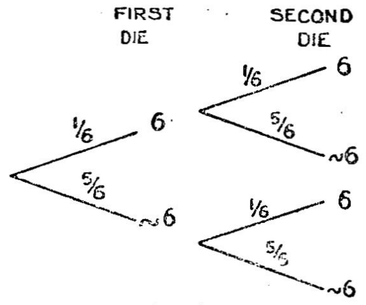


Fig. 4-3.

Event	Probability
(6, 6)	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
(6, ~ 6)	$\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$
(~ 6, 6)	$\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$
(~ 6, ~ 6)	$\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$P(6, 6) = P(6 \text{ in the first die}) \times P(6 \text{ in the 2nd die})$

$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$P(6, \sim 6) = P(6 \text{ in the first die}) \times P(\sim 6 \text{ in the 2nd die})$

$= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

$P(\sim 6, 6) = P(\sim 6 \text{ in the first die}) \times P(6 \text{ in the 2nd die})$

$= \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

$P(\sim 6, \sim 6) = P(\sim 6 \text{ in the first die}) \times P(\sim 6 \text{ in the 2nd die})$

$= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

Hence

(i) $P(\text{two 6s}) = P(6, 6) = \frac{1}{36}$

(ii) $P(\text{one 6}) = P(6, \sim 6) + P(\sim 6, 6)$

$= \frac{5}{36} + \frac{5}{36} = \frac{10}{36} = \frac{5}{18}$

(iii) $P(\text{two} \sim 6\text{s}) = P(\sim 6 \sim 6) = \frac{25}{36}$

Example 2 :

The probability that the crackers manufactured in a factory are defective, is $\frac{1}{10}$. Selvi bought 3 crackers. What is the probability that all the three crackers are good ?

Solution :

(In this set we need not draw the tree diagram completely. The relevant portion alone may be drawn to find the solution.)

The probability that a cracker is good $= 1 - \frac{1}{10} = \frac{9}{10}$

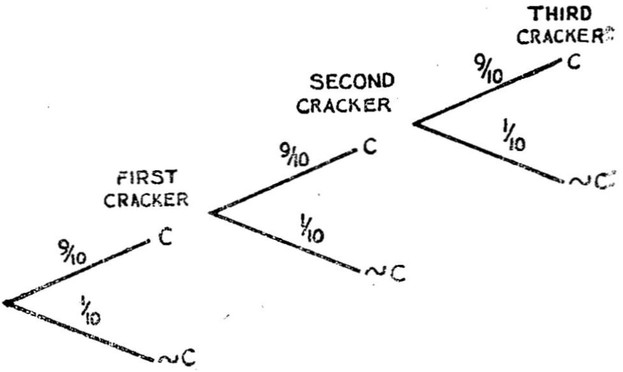


Fig. 4-4

$$P\text{ (three crackers are good)} = P\text{ (First is good)} \times P\text{ (2nd is good)} \times P\text{ (3rd is good)}$$

$$= \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} = \frac{729}{1000}$$

Example 3 :

A and B apply for a job. The probablility that A will be selected is $\frac{1}{7}$ and that B will be selected is $\frac{1}{5}$. What is the probability (i) that both of them will be selected and (ii) that both of them will not be selected ?

Solution :

(Without drawing the tree-diagram we can find the solution using the multiplication theorem of probability)

(i) The events of A and B being selected are independent events

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B) = \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$$

(ii) The events that both of them are not selected are also independent events.

$$\therefore \quad P(\sim A \cap \sim B) = P(\sim A)\, P(\sim B)$$
$$= [1 - P(A)]\,[1 - P(B)]$$
$$= \left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{5}\right)$$
$$= \frac{6}{7} \times \frac{4}{5}$$
$$= \frac{24}{35}$$

Exercise 4.4-5

1. In a packet of flower-seeds, the probability that a seed grows into a plant is $\frac{4}{5}$ and the probability of a plant yielding flowers is $\frac{5}{6}$. What is the probability of a selected seed growing into a plant and yielding flowers ?

2. The probability that A solves the problems in a book is 0·75 and that B solves them is 0·60. If a problem is chosen at random, what is the probability (i) that both of them solve it and (ii) that either A or B solves it ?

3. A and B are independent events. If $P(A) = 0{\cdot}4$ and $P(B) = 0{\cdot}5$. Find

   (i) $P(A \cap B)$ (ii) $P(A \cap \sim B)$

   (iii) $P(\sim A \cap B)$ (iv) $P(\sim A \cap \sim B)$

   (v) $P(A \cup B)$

4. The probability of a man being alive, for 25 years, is $\frac{3}{5}$. The probability of his wife being alive for 25 years is $\frac{2}{3}$. In 25 years, what is the probability that (i) both of them are alive, (ii) only the husband is alive, (iii) only the wife is alive and (iv) only one of the two is alive ?

ANSWERS

(1) $\frac{2}{3}$ (2) (i) 0·45 (ii) 0·9

(3) (i) 0·2 (ii) 0·2 (iii) 0·3 (iv) 0·3 (v) 0·7

(4) (i) $\frac{2}{5}$ (ii) $\frac{1}{5}$ (iii) $\frac{4}{15}$ (iv) $\frac{13}{15}$

4·4·6. Probability : Multiplication Theorem: Dependent Events :

Let us examine the event of taking 2 marbles out of a bag containing 2 red and 4 blue marbles.

When the first marble is taken, the total number of marbles is 6.

$$\therefore \quad P(R) = \frac{2}{6}; \quad P(B) = \frac{4}{6}$$

When the 2nd marble is taken, the number of marbles in the bag is only 5. If the marble taken first is red, 4 of the remaining 5 marbles is blue and 1 is red.

$$\therefore \quad P(R) = \frac{1}{5}; \quad P(B) = \frac{3}{5}$$

Therefore, when the second marble is taken, the probability depends on the marble taken first. These are called **dependent events.**

Example 1 :

There are 4 blue balls and 5 red balls in a bag. A ball is taken first and then without putting it into the bag, another ball is taken. What is the probability that (i) both of them are blue, (ii) one is blue and the other is red and (iii) both of them are red?

Solution :

$$P(B, B) = \frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$$

$$P(B, R) = \frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$$

$$P(R, B) = \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$$

$$P(R, R) = \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$$

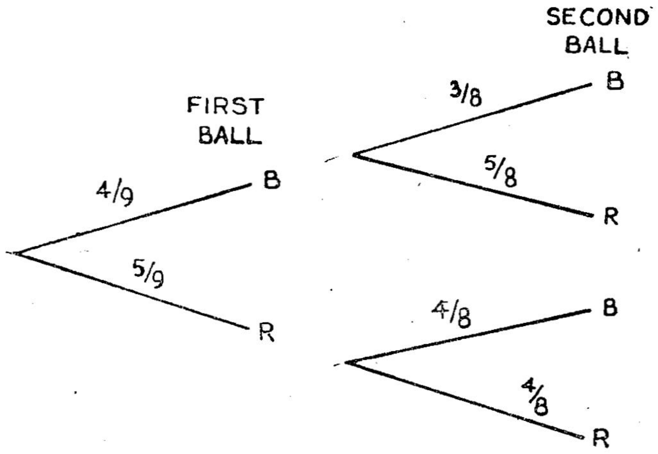


Fig. 4-5

(i) For both of them to be blue: $P(B, B) = \frac{12}{72} = \frac{1}{6}$

(ii) For one to be blue and the other red:

$$P(B, R) + P(R, B)$$
$$= \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

(iii) For both of them to be red :

$$P(R, R) = \frac{20}{72} = \frac{5}{18}$$

Example 2 :

There are 100 transistors in a box. 20 of them are defective. At random two transistors are taken one by one consecutively. What is the probability that (i) both of them are

good (ii) both of them are defective and (iii) one of them is defective and the other is good ?

Solution :

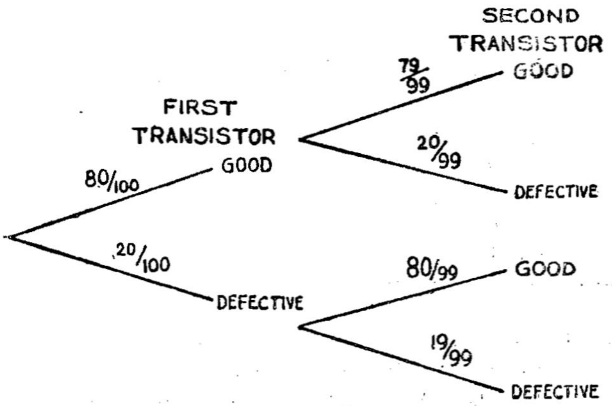


Fig. 4·6

Event	Probability
2 good transistors	$\frac{4}{5} \times \frac{79}{99} = \frac{316}{495}$
Good and defective transistors	$\frac{4}{5} \times \frac{20}{99} = \frac{80}{495}$
Defective and good transistors	$\frac{1}{5} \times \frac{80}{99} = \frac{80}{495}$
2 defective transistors	$\frac{1}{5} \times \frac{19}{99} = \frac{19}{495}$

(i) For 2 transistors to be good,

the probability is $= \frac{316}{495}$

(ii) For 2 transistors to be defective,

the probability is $= \frac{19}{495}$

(iii) For one to be good and the other to be defective,

the probability is $= \frac{80}{495} + \frac{80}{495} = \frac{160}{495} = \frac{32}{99}$

Exercise 4·4·6

1. 10 out of 12 eggs in a basket are good. At random 2 eggs are taken one by one. What is the probability that (i) both of them are good and (ii) both of them are bad ?

2. Only one of the 100 mangoes in a basket is rotten. If two mangoes are taken out one by one at random, what is the probability that any one of them is good ?

3. There are 7 red pens and 3 blue ones in a box. A takes at random one pen first and then B takes another. Out of the pens taken by both of them, what is the probability that (i) both pens are of the same colour and (ii) both of them are of different colours ?

4. There are 8 green and 4 yellow marbles in a bag. At random, two marbles are taken one after another, consecutively. What is the probability that the first marble is yellow and the second is green ?

ANSWERS

1. (i) $\frac{15}{22}$ (ii) $\frac{1}{66}$
2. $\frac{1}{50}$
3. (i) $\frac{8}{15}$ (ii) $\frac{7}{15}$
4. $\frac{8}{33}$

REVIEW

1. In a frequency table the data will be grouped into classes and the number of items in a class is the frequency of the class.

2. (a) The Arithmetic mean of n data $\bar{x} = \frac{\Sigma x}{n}$

   If A is the Assumed Mean, the Arithmetic Mean

$$\bar{x} = A + \frac{\Sigma d}{n}; \quad [d = x - A]$$

(b) In a frequency table the Arithmetic Mean

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}; \; [\Sigma f = n]$$

$$\text{or } \bar{x} = A + \frac{\Sigma fd}{n}; \quad A \text{ is the assumed mean } \; d = x - A$$

3. The Median of n values $x_1, x_2, \ldots\ldots, x_n$

(a) If n is an odd number such that $n = 2k - 1$

then $M = x_k$

(b) If n is an even number such that $n = 2k$

$$\text{then } M = \frac{x_k + x_{k+1}}{2}$$

4. In the classification of data into a frequency table, the class which has the highest frequency is called the **Mode.**

5. The difference between the highest value and the lowest value of the data is the **Range.**

6. (a) The **Standard Deviation** of Discrete series

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}}; \quad [d = x - \bar{x}]$$

(b) The Standard Deviation in a frequency table

$$\sigma = \sqrt{\frac{\Sigma fd^2}{n}}; \quad [d = x - \bar{x}]$$

7. (a) In an experiment having equal chances, the set of all outcomes is called the Sample space, S. The subset of the sample space is called the event, E.

The probability of an event $P(E) = \dfrac{n(E)}{n(S)}$

(b) The probability of a sure event is 1.

The probability of an event which does not occur at all is 0.

(c) The events, which are such that if one occurs the other does not occur are called **mutually exclusive events.**

If A and A' are mutually exclusive events,

$$P(A) + P(A') = 1$$

or $P(A') = 1 - P(A)$

Further $P(A \cap A') = 0$

(d) If A and B are not mutually exclusive events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(e) If the occurrence of one event does not affect the probability of another event, they are called **independent events.**

If A and B are independent events,

$$P(A \cap B) = P(A) + P(B) - P(A \cap B)$$

(f) The tree diagram will be useful in calculating the probability of an event which repeatedly occur in an experiment.

SELF ASSESSMENT TEST

The following are the weights of 20 mangoes taken from a basket (in gm.)

106	123	90	113	99
107	92	107	98	128
76	84	70	86	82
187	119	80	81	90

Form a frequency table, choosing proper limits of class intervals for these data.

2. Find the mean for the data in the following frequency table.

x	f
4·5 — 5·5	6
5·5 — 6·5	8
6·5 — 7 5	12
7·5 — 8·5	2
8 5 — 9 5	2
	$\Sigma f = 30$

3. Find the standard deviation of the following frequency table.

x	2	6	10	14	
f	4	8	2	1	$\Sigma f = 15$

4. There are 6 blue balls, 4 green balls and 5 red balls in a bag. If a ball is taken at random from the bag what is the probability (i) that it is either green or red and (ii) that it is not red or green ?

5. A die is thrown twice. What is the probability that the number on the die in each throw is (i) > 5; (ii) < 5.

ANSWRES

(2) 6·53 (3) 3·24

(4) (i) $\frac{13}{5}$ (ii) $\frac{2}{5}$ (5) (i) $\frac{1}{9}$ (ii) $\frac{25}{36}$

NOT FOR EXAMINATION

MATHEMATICS CLUB — ACTIVITY IV

Take 5 cardboard pieces containing numbers written as in the diagram. Note that the numbers are written on both sides of the 5th cardboard. With the help of these you can find the number which a person has in his mind. Showing the cards one by one, you ask the person whether the number he has in his mind finds a place in that card. If he says 'Yes', place the card on the table such that the word 'Yes' is kept open. If he says 'No', place the card such that the word 'No' is kept

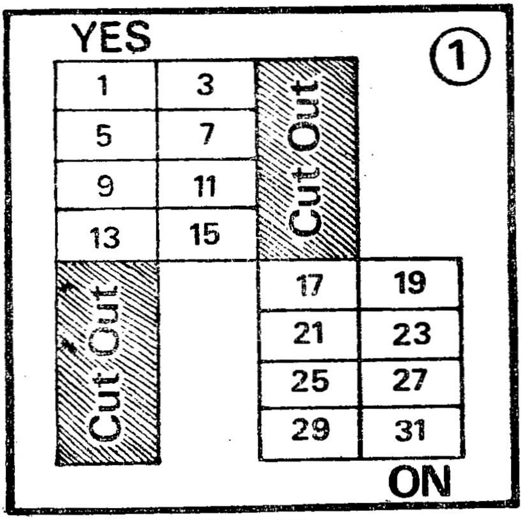


Fig. 4-7.

open. After all the five cards have been placed on the table, pile them one upon the other and show it to your friend. He will see the number which he had in his mind.

Let us examine how the numbers on the card are formed. On the first card we have just odd numbers. If we write the numbers as powers of 2, in a serial order, if the coefficient of 2^0 is 1, then that number will be in the first card. If the coefficient of 2^1 is 1, then that number will be found in the second card. If the coefficient of 2^2 is 1, then it will be found in the 3rd card and so on.

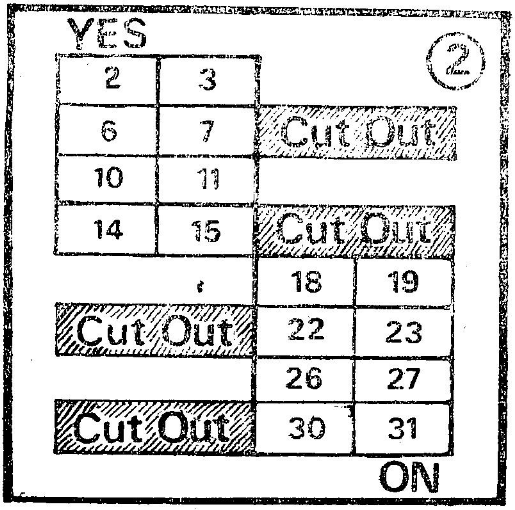

YES
2
2
3
6
7
Cut Out
10
11
14
15
Cut Out
18
19
Cut Out
22
23
26
27
Cut Out
30
31
ON

Fig. 4-8.

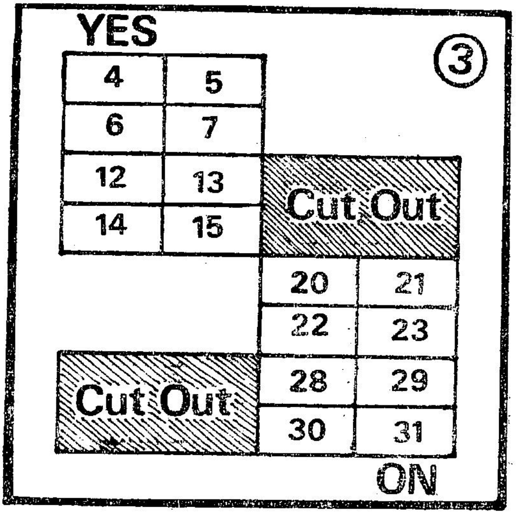

YES
3
4
5
6
7
12
13
Cut Out
14
15
20
21
22
23
Cut Out
28
29
30
31
ON

Fig. 4-9.

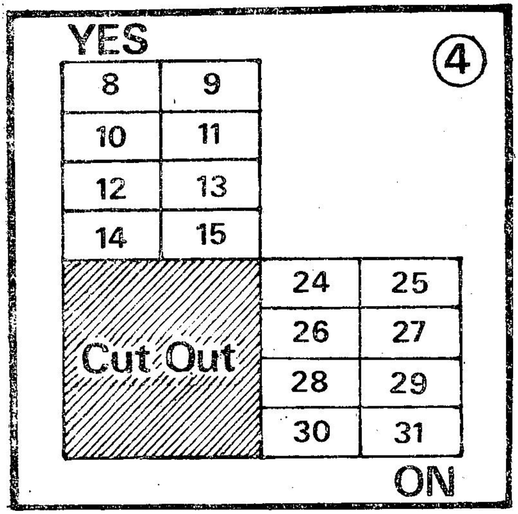


Fig. 4-10.

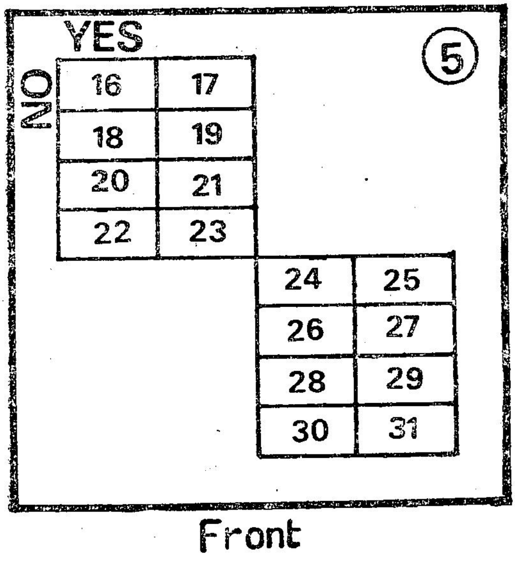
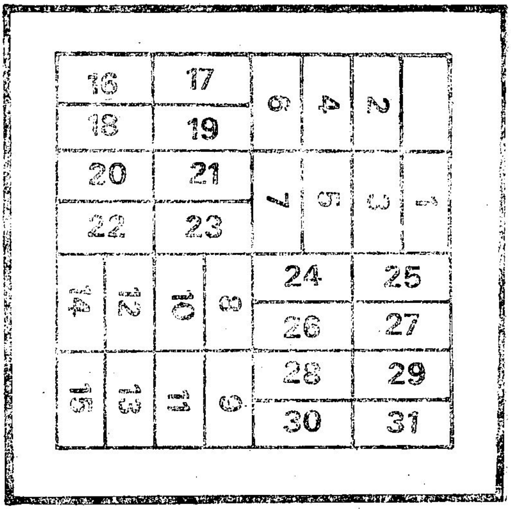


Fig. 4-11.



Back

Fig. 4-12.

Number	Power 0	Power 1	Power 2	Power 3	Power 4	Card having the number
1	$1\cdot 2^{0}$					1
2	$0\cdot 2^{0}$	$1\cdot 2^{1}$				2
3	$1\cdot 2^{0}$	$1\cdot 2^{1}$				1, 2
4	$0\cdot 2^{0}$	$0\cdot 2^{1}$	$1\cdot 2^{2}$			3
5	$1\cdot 2^{0}$	$0\cdot 2^{1}$	$1\cdot 2^{2}$			1, 3
6	$0\cdot 2^{0}$	$1\cdot 2^{1}$	$1\cdot 2^{2}$			2, 3
7	$1\cdot 2^{0}$	$1\cdot 2^{1}$	$1\cdot 2^{2}$			1, 2, 3

The responses 'NO' and 'YES' are equivalent to the numbers 1, 0 respectively of base 2. In the same manner, prepare cardboard bearing the numbers upto 63 and upto 127.

5. MENSURATION

5·1·1. Prism : Surface Area

Recall :

(a) Surface Area of a right prism

= Perimeter of the cross—section of the prism × height of the prism

$S = P \times h$ square units

(P = Perimeter ; h = height)

(b) Total Surface Area of a right prism :

$T = Ph + 2A$ Square units where A stands for the area of the Cross - section of the prism, P for perimeter and h for height.

Example :

The cross - section of a prism is an equilateral triangle of side 5 cm. Its height is 12 cm. Find its surface area and total surface area. ($\sqrt{3} \simeq 1{\cdot}732$)

Solution :

Perimeter of the equilateral triangle = 15 cm

Surface Area of the prism, S $= P \times h$

$= 15 \times 12 \text{ cm}^2$

$= 180 \text{ cm}^2$

Area of the equilateral triangle, A $= \dfrac{\sqrt{3}}{4} a^2$

$= \dfrac{5 \times 5 \times 1{\cdot}732}{4}$

$= 10{\cdot}825 \text{ cm}^2$

Total Surface Area of the prism $= S + 2A$

$= 180 + (2 \times 10{\cdot}825)\ cm^2$

$= 180 + 21{\cdot}65\ cm^2$

$= 201{\cdot}65\ cm^2$

Exercise 5-1·1

Find the surface area and the total surface area of the prisms whose measurements are given below.

	Cross - section	Height
1.	A square of side 6 cm	10 cm
2.	A square of side 8 cm	20 cm
3.	A rectangle 15 cm × 10 cm	12 cm
4.	An equilateral triangle of side 6 cm	15 cm
5.	A regular hexagon of side 10 cm	12 cm
6.	A right angled triangle, the sides containing the right angle being 8 cm , 15 cm	20 cm

ANSWERS

(1) 240 cm², 312 cm² (2) 640 cm², 768 cm²

(3) 1800 cm², 2'00 cm² (4) 270 cm², 301·176 cm²

(5) 720 cm², 1239·6 cm² (6) 800 cm², 920 cm²

5-1.2 Prisms : Volume

Recall :

Volume of a prism

= Area of the cross—section of the prism × height of the prism

$V = A \times h$ cubic units, where A stands for the area of the cross-section and h for height.

Example :

The height of a hexagonal prism is 10 cm. The side of the hexagon is 4 cm. Find its volume. ($\sqrt{3} \simeq 1{\cdot}732$).

Solution :

Area of the cross-section = Area of the regular hexagon

$$A = \frac{6a^2\sqrt{3}}{4}$$

$$= 6 \times 4 \times 4 \times \frac{1.732}{4}$$

$$= 41{\cdot}568 \text{ cm}^2$$

Volume of the prism $= A \times h$

$$= 41{\cdot}568 \times 10 \text{ cm}^3$$

$$= 415{\cdot}68 \text{ cm}^3.$$

Exercise 5-1·2

Find the volume of the prisms with the measurements given below.

	Cross-section	Height
1.	An equilateral triangle of side 7 cm	14 cm
2.	A regular hexagon of side 6 cm	60 cm
3.	A rectangle 10×16 cm	15 cm
4.	A square of side 8 cm	14·5 cm
5.	A right angled triangle, the sides containing the right angle being 7 cm, 10 cm	9 cm

ANSWERS

(1) 297·038 cm³ (2) 1870·56 cm³

(3) 1800 cm³ (4) 928 cm³

(5) 315 cm³

5-1·3 Cylinder : Curved Surface Area

Recall :

1. The curved surface area of a cylinder is $P \times h$ sq. units.

$$S = 2\pi r \times h \text{ sq. units}$$

$$\therefore \; S = 2\pi rh$$

2. The total surface area of a cylinder is. $Ph + 2A$ sq. units

$$= 2\pi rh + 2\pi r^2 \text{ sq units}$$
$$= 2\pi r\,(h + r) \text{ sq. units}$$
$$T = 2\pi r\,(h + r)$$

(r = radius h = height)

Example:

The radius of a cylinder is 12 cm. Its height is 30 cm. Find its curved surface area and the total surface area.

Solution :

1. The curved surface area $= 2\pi rh$

$$= 2 \times \pi \times 12 \times 30 \text{ cm}^2$$
$$= 720\pi \text{ cm}^2$$

2. The total surface area of the cylinder

$$= 2\pi r\,(h + r)$$
$$= 2 \times \pi \times 12 \times 42 \text{ cm}^2$$
$$= 1008\pi \text{ cm}^2$$

Exercise 5-1·3

[Note: Use $\pi = \frac{22}{7}$]

1. Find the volume of the cylinder with the measurements given below :

Radius (r)	Height (h)
(a) 21 cm	20 cm
(b) 28 cm	15 cm
(c) 17·5 cm	12 cm

2. The area of the base of a cylinder is 154 cm Its height is 16 cm. Find its curved surface area.

3. The perimeter of the corrs-section of a cylinder is 100 cm. Its height is 10 cm. Find its curved surface area.

4 The diameter of a cylinder is 21 cm. Its height is 22 cm. Find its curved surface area.

5. The radii of two cylinders are in the ratio 3 : 4 while their heights are in the ratio 5 : 6. Find the ratio of the curved surface areas of the two cylinders.

6. The curved surface area of a cylinder is 1760 cm^2. Its radius is 14 cm. Find its height.

7. The total surface area of a cylinder is 748 cm. The radius is 7 cm. Find its height.

ANSWERS

(1) (a) 2640 cm^2, 5412 cm^2

(b) 2640 cm^2, 7568 cm^2

(c) 1320 cm^2, 3245 cm^2

(2) 704 cm^2 (3) 1000 cm^2 (4) 1452 cm^2

(5) 5 : 8 (6) 20 cm (7) 10 cm

5-1-4. Cylinder : Volume

Recall :

Volume of a cylinder $= \pi r^2 h$ cubic units

(r = radius, h = height)

Example :

The diameter of a cylinder is 21 cm. Its height is 12 cm. Find its volume. $\left(\pi = \frac{22}{7}\right)$

Solution :

Volume of a cylinder $= \pi r^2 h$

$$= \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 12 \text{ cm}^3$$

$$= 4158 \text{ cm}^3$$

Exercise 5-1-4

[Note : Use $\pi = \frac{22}{7}$]

1. Find the volume of the cylinder with the following measurements.

	radius (r)	height (h)
(a)	7 cm	10 cm
(b)	28 cm	30 cm
(c)	10·5 cm	15 cm

2. The diameter of a cylinder is 35 cm. Its height is 18 cm. Find its volume.

3. Find the ratio of the volumes of the two cylinders whose radii are in the ratio 5 : 6, and whose heights are in the ratio 3 : 4.

4. The ratio of the radii of two cylinders is 3 : 7. Their heights are also in the same ratio. Find the ratio of their volumes.

5. Two cylinders are equal in height and their radii are in the ratio 4 : 5. Find the ratio of their volumes.

6. The area of the base of a cylinder is 80 cm^2. Its height is 10 cm. Find its volume.

7. The perimeter of the cross - section of a cylinder is 264 cm. Its height is 40 cm. Find its volume.

8. The volume of a cylinder is 6600 cm^3 Its height is 21 cm. Find its radius.

9. The diameter of a cylindrical test tube is 1 cm. The height of the mercury column in the tube is 20 cm. Find the weight of the mercury in the test tube, given that the density of mercury is 13·6.

10. A rectangular cardboard of dimensions 22 cm × 12 cm is rolled to form a cylinder by joining the smaller sides. Find the volume of the cylinder thus formed.

11. The diameter of a cylindrical pipe is 7 cm. Water flows through the pipe at a speed of 2·5 km per hour. Calculate the volume of water that flows in one hour.

ANSWERS

(1)	(a) 1540 cm^3	(b)	73920 cm^3	(c)	5197 5 cm^3
(2)	17325 cm^3	(3)	25 : 48	(4)	27 : 343
(5)	16 : 25	(6)	800 cm^3	(7)	221760 cm^3
(8)	10 cm	(9)	213·52 gm	(10)	462 cm^3
(11)	9625000 cm^3				

5-2-1. Cone : Introduction

The cone is bounded by a flat circular plane and a curved surface with a pointed vertex. The circular portion is known as the base of the cone. The line joining the vertex and any

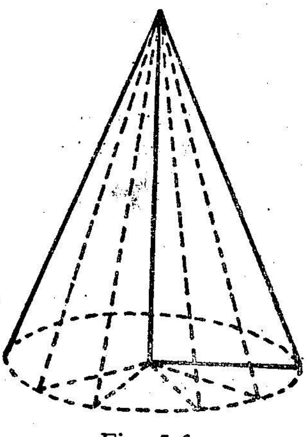


Fig. 5-1

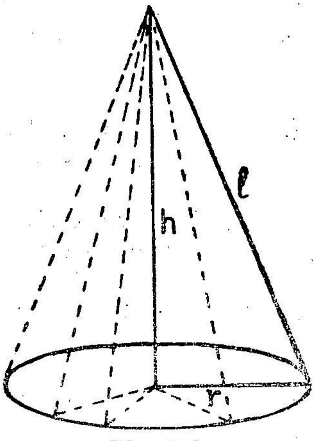


Fig. 5-2

point on the circumference of the base lies on the curved surface. If a right angled triangle is rotated with one of the sides containing the right angle as axis, it generates a cone.

The slant height of the cone l is the distance between the vertex of the cone and any point on the circumference of the base. The line joining the centre of the base and the vertex is perpendicular to the plane of the circular base. This is the height of the cone, h.

What will be the radius, height and slant height of the cone when a right-angled triangle is rotated with one of its sides containing the right angle as axis ?

Hypotenuse	= Slant height
The side which forms the axis	= height
The other side	= radius of the base

In fig. 5·3.

OA = height (h)

AP = radius (r)

OP = slant height (l)

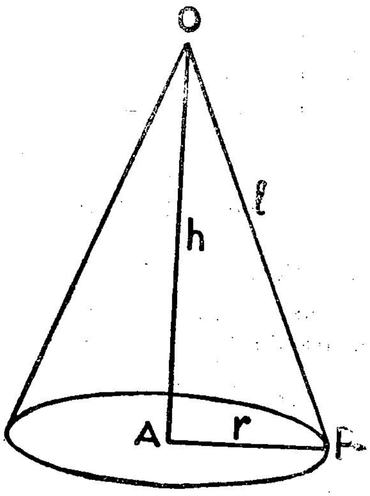


Fig. 5·3.

By Pythagoras theorem,

$$l^2 = r^2 + h^2$$

$$l = \sqrt{r^2 + h^2}$$

$$h = \sqrt{l^2 - r^2}$$

$$r = \sqrt{l^2 - h^2}$$

Example 1 :

The radius of a cone is 5 cm. Its height is 12 cm. Find its slant height.

Solution :

$$l = \sqrt{r^2 + h^2}$$

$$= \sqrt{5^2 + 12^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

$$= 13$$

Slant height = 13 cm

Exercise 5-2·1

Complete the following table.

S. No.	Radius (r)	Height (h)	Slant height (l)
1	3 cm	4 cm	—
2	8 cm	—	17 cm
3	—	24 cm	25 cm
4	9 cm	40 cm	—
5	20 cm	—	29 cm

ANSWERS

(1) 5 cm (2) 15 cm (3) 7 cm (4) 41 cm (5) 21 cm

5-2.2 Cone : Curved surface area

Take a sector of a circle. Join together its straight edges. The resulting shape is a cone.

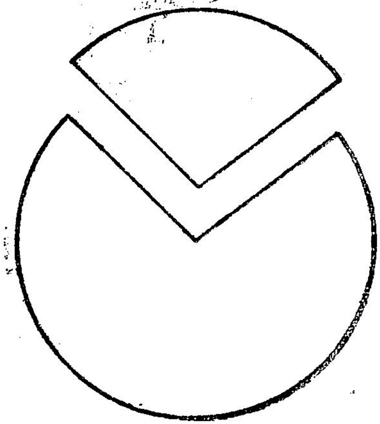


Fig. 5·4

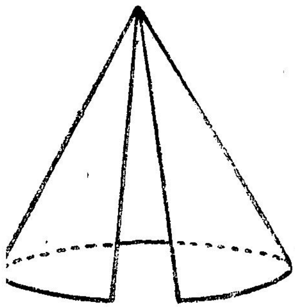


Fig. 5·5

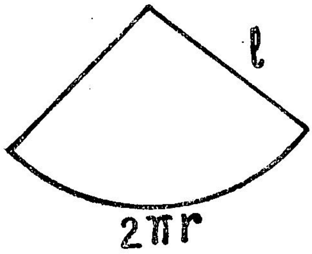


Fig. 5·6

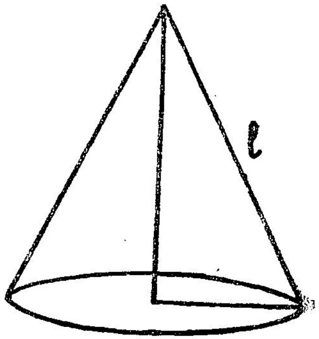


Fig. 5·7

Relation between the sector and the coen :

Area of the sector = Curved surface area of the cone.

Length of the arc of the sector = Perimeter of the circular base of the cone.

Radius of the sector = Slant height of the cone.

Hence we derive the formula for the curved surface area thus.

Curved surface area = Area of the sector

$$= \frac{\text{Length of the area of the sector} \times \text{radius}}{2}$$

$$= \frac{\text{Perimeter of the circular base} \times \text{slant height}}{2}$$

$$= \frac{2\pi r \times l}{2}$$

$= \pi r l$ sq. units

Total surface area of the cone

= Curved surface area + area of the circular base.

$= \pi r l + \pi r^2$

Example :

The radius of a cone is 8 cm. Its height is 15 cm. Find its curved surface area and the total surface area.

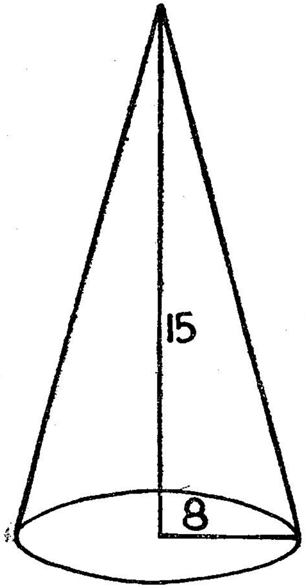


Fig. 5-8

Solution :

$$l = \sqrt{h^2 + r^2}$$

$$= \sqrt{15^2 + 8^2}$$

$$= \sqrt{225 + 64}$$

$$= \sqrt{289}$$

$$= 17$$

Curved surface area

$$= \pi r l$$

$$= \pi \times 8 \times 17 \text{ cm}^2$$

$$= 136\pi \text{ cm}^2$$

Total surface area $= \pi r (l + r)$

$$= \pi \times 8 \times (17 + 8) \text{ cm}^2$$

$$= \pi \times 8 \times 25 \text{ cm}^2$$

$$= 200\pi \text{ cm}^2$$

Exercise 5-2·2

1. Find the curved surface area and the total surface area of the cones with the following measurements.

   (a) radius 20 cm, slant height 30 cm.

   (b) radius 7 cm, height 24 cm

   (c) radius 9 cm, height 40 cm

   (d) height 40 cm, slant height 50 cm

2. The radii of two cones are in the ratio 3 : 4 and their slant heights are in the ratio 5 : 6. Find the ratio of the curved surface area of the cones.

3. The perimeter of the base of a cone is 48 cm. Its slant height is 20 cm. Find its curved surface area.

4. The area of the base of a cone is 36π cm². The slant height of the cone is 8 cm. Find its total surface area.

5. The curved surface area of a cone is $204\frac{2}{7}$ cm². Its radius is 5 cm. Find its height. $\left(\pi = \frac{22}{7}\right)$

6. The total surface area of a cone is 4400 cm². Its diameter is 40 cm. Find its height. $\left(\pi = \frac{22}{7}\right)$

7. The ratio between the radius and slant height of a cone is 2 : 5. Its curved surface area is 160π cm². Find its height.

ANSWERS

1. (a) 600π cm²; 1000π cm²
   (b) 175π cm²; 224π cm²
   (c) 369π cm²; 450π cm²
   (d) 1500π cm²; 2400π cm²

2. 5 : 8; 3. 480π cm²; 4. 96π cm²; 5. 12 cm
6. $10\sqrt{21}$ cm 7. $4\sqrt{21}$ cm

5·2·3 Cone : Volume :

Take a hollow cone and a hollow cylinder of the same internal diameter and height. Fill the cone with water and pour it into the cylinder. Do this thrice. It will be found that the cylinder is just completely filled.

From this we find that the volume of the cylinder is thrice the volume of the cone.

Volume of the cylinder $= \pi r^2 h$

Volume of the cone of the same radius and height

$$= \frac{1}{3} \pi r^2 h$$

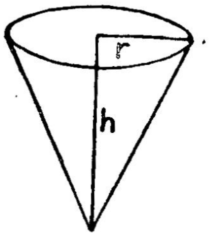


Fig. 5-9.

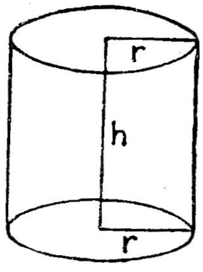


Fig. 5-10.

Example :

The radius of a cone is 9 cm. Its slant height is 15 cm. What is its volume ?

Solution :

$$h = \sqrt{l^2 - r^2}$$
$$= \sqrt{15^2 - 9^2}$$
$$= \sqrt{225 - 81}$$
$$= \sqrt{144}$$
$$= 12 \text{ cm.}$$

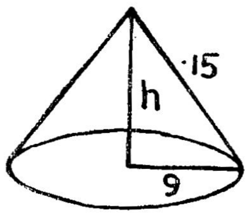


Fig 5-11.

$$V = \frac{1}{3} \pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 9 \times 9 \times 12 \text{ cm}^3$$
$$= 324\pi \text{ cm}^3$$

Exercise 5-2·3

1. Find the volume of the cone with the measurements given below.

   (a) Radius 8 cm, height 12 cm

   (b) Radius 10 cm, height 21 cm

(c) Radius 8 cm, slant height 17 cm

(d) Radius 5 cm, slant height 13 cm

(e) Slant height 41 cm, height 40 cm

2. The ratio of the radii of two cones is 3 : 5 ; the ratio of their heights is 4 : 7. Find the ratio of their volumes.

3. The radii and the heights of two cones are both in the ratio 5 : 6. Find the ratio of their volumes.

4. The ratio of radii of two cones of equal height is 1 : 3. Find the ratio of their volumes.

5. Two cones of equal radii have their heights in the ratio 10 : 11. Find the ratio of the volumes of the cones.

6. The area of the base of a cone is 300 cm^2. Its height is 10 cm. Find its volume.

7. The perimeter of the base of a cone is 40π cm. Its slant height is 29 cm. Find its volume.

8. The curved surface area of a cone is 39π cm^2. Its slant height is 6·5 cm. Find its volume.

9. The total surface area of a cone is 56π cm^2. Its radius is 3·5 cm. Find its volume.

10. The area of the base of a conical tent is $201\frac{1}{7}$ m^2. Its lateral surface area is $427\frac{3}{7}$ m^2. How many people can be seated in it if each requires $22\frac{6}{7}$ m^3 air ? $\left(\pi = \frac{22}{7}\right)$

11. Find the volume and curved surface area of the cone generated by rotating a right angled triangle, the sides containing the right angle being 9 cm and 12 cm, and the axis being the side of 9 cm.

12. The radius of the base of a cone is 7 cm. Its vertical angle is 60°. Find its volume and curved surface area.

13. The volume of a cone is 320π cm³. Its radius is 8 cm. Find its curved surface area and total surface area.

ANSWERS

(1) (a) 256π cm³ (b) 700π cm³ (c) 320π cm³ (d) 100π cm³ (e) 1080π cm³

(2) 36 : 175 (3) 125 : 216 (4) 1 : 9 (5) 10 : 11

(6) 1000 cm³ (7) 2800π cm³ (8) 30π cm³ (9) 49π cm³

(10) 44 (11) 432π cm³, 180π cm²

(12) $\frac{343}{\sqrt{3}}\pi$ cm³, 98π cm² (13) 120π cm², 184π cm²

5-2·4 The Sector of a Circle and Cone

Example :

A sector with angle measure 216° and radius 35 cm is taken and folded into a cone by joining the straight edges. What is the volume of the cone thus formed? $\left(\pi \doteq \frac{22}{7}\right)$

Solution :

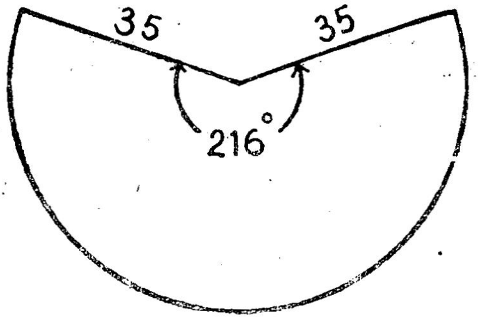


Fig. 5-12.

Length of the arc of the sector

= Perimeter of the base of the cone

$$\text{Length of the arc} = \frac{x}{360} \times 2\pi r$$

$$= \frac{216}{360} \times 2 \times \frac{22}{7} \times 35 \text{ cm}$$

$$= 132 \text{ cm}$$

Perimeter of the base of the cone $= 2\pi r$

$$2\pi r = 132$$

$$2 \times \frac{22}{7} \times r = 132$$

$$r = 132 \times \frac{1}{2} \times \frac{7}{22}$$

radius of the cone = 21 cm

radius of the sector = slant height of the cone

$$= 35 \text{ cm}$$

$$h^2 = 35^2 - 21^2$$

$$= 1225 - 441$$

$$= 784$$

$$h = \sqrt{784}$$

$$= 28 \text{ cm.}$$

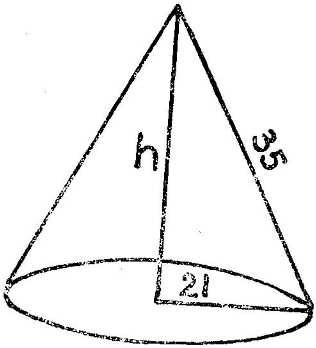


Fig. 5-13

Volume of the cone

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \frac{22}{7} \times 21 \times 21 \times 28 \text{ cm}^3$$

$$= 12936 \text{ cm}^3$$

Exercise 5-2·4

1. From a circular plate of radius 18 cm, a minor sector of central angle 140° is cut and folded into a cone by joining the radiis. Find the radius of the cone.

2. In question 1, if the major sector left out is folded into a cone, what is its height ?

3. A sector of radius 10 cm and central angle 288° is made into a cone by bringing the two radii together. Find the volume of the cone.

4. A sector of central angle $100\frac{4^\circ}{5}$ and radius 25 cm is made into a cone by joining the two radii. Find its volume.

ANSWERS

(3) $402\frac{2}{7}$ cm³ (4) 1232 cm³

5-2·5. Spheres : Volume and Surface area

Volume of the sphere $V = \frac{4}{3}\pi r^3$ cubic units.

Surface area of the sphere $= 4\pi r^2$ square units

(r — radius of the sphere)

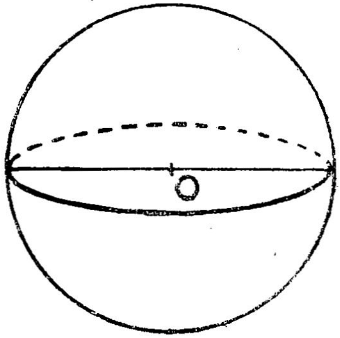


Fig. 5-14

Example 1 :

The radius of a sphere is 12 cm. Find its volume.

Solution :

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi \times 12 \times 12 \times 12$$

$$= 2304\pi \text{ cm}^3$$

Example : 2

The radius of a sphere is 7 cm. Find its surface area. $\left(\pi = \frac{22}{7}\right)$

Solution :

$$\begin{aligned} \text{Surface area of the sphere} &= 4\pi r^2 \\ &= 4 \times \pi \times 7 \times 7 \text{ cm}^2 \\ &= 616 \text{ cm}^2 \end{aligned}$$

Exercise 5-2-5

[Note : Use $\pi = \frac{22}{7}$]

1. The radius of a sphere is 14 cm. Find its volume.
2. The diameter of a sphere is 10 cm. What is its volume ?
3. The ratio of the radii of two spheres is 4 : 5. Find the ratio of their volumes.
4. The ratio of the volumes of two spheres is 216 : 343. Find the ratio of their radii.
5. The diameter of a stone ball is 7 cm. 1 cc of stone weighs 2·7 gm. What is the weight of the ball ?
6. Calculate the change in the volume of a sphere when its radius is increased by half.
7. Calculate the surface area of a sphere whose radius is 10 cm.
8. The diameter of a sphere is 21 cm. Find its surface area.
9. The ratio of the radii of two spheres is 7 : 8. What is the ratio of their surface areas ?

10. The surface area of a sphere is 616 cm^2 What is its volume? $\left(\pi \simeq \frac{22}{7}\right)$

11. The volume of a sphere is 38808 cm^3. Calculate its surface area? $\left(\pi \simeq \frac{22}{7}\right)$

ANSWERS

(1) $11498\frac{2}{3}$ cm^3	(2) 523·3 cm^3	(3) 64 : 125
(4) 6 : 7	(5) 485·1 cm^3	(6) $3\frac{3}{8}$ times
(7) 400π cm^2	(8) 1386 cm^2	(9) 49 : 64
(10) $1437\frac{1}{3}$ cm^3	(11) 5544 cm^2	

5-2.6 Hemisphere

Volume of the hemisphere $= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$$= \frac{2}{3}\pi r^3$$

Curved surface area of the hemisphere

$$= \frac{1}{2} \times 4\pi r^2$$

$$= 2\pi r^2$$

Total surface area of the hemisphere

$$= 2\pi r^2 + \pi r^2$$

$$= 3\pi r^2$$

Example :

The radius of a hemisphere is 6 cm. Find its volume, surface area and total surface area.

Solution

Volume of the hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \pi \times 6 \times 6 \times 6 \text{ cm}^3$$

$$= 144\pi \text{cm}^3$$

Curved surface area of the hemisphere

$$= 2\pi r^2$$
$$= 2\times\pi\times 6\times 6 \text{ cm}^2$$
$$= 72\pi \text{ cm}^2$$

Total surface area of the hemisphere

$$= 3\pi r^2$$
$$= 3\times\pi\times 6\times 6 \text{ cm}^2$$
$$= 108\pi \text{ cm}^2$$

Exercise 5·2·6

1. Find the volume, curved surface area and total surface area of a sphere with radius 3·5 cm. $\left(\pi = \frac{22}{7}\right)$
2. The diameter of a hemisphere is 11 cm. Calculate its volume, curved surface area and total surface area.
3. The ratio of the diameters of two spheres is 5 : 7. Find the ratios of their volumes and their curved surface areas.

ANSWERS

(1) 89·83 cm^3, 77 cm^2, 115·5 cm^2

(2) 110·9π cm^3, 60·5π cm^2, 90·75π cm^2

(3) 125 : 343, 25 : 49

5·2·7 Cone and Cylinder

Example :

A tent in an exhibition is in the form of a cone mounted on a cylinder. The height and radius of the cylinder are 10 m and 12 m respectively. The total height of the tent is 15 m. Find the area of the canvas that is required for the tent.

Solution :

Radius of the cylinder = 12 m

Height of the cylinder = 10 m

Curved surface area of the cylinder $= 2\pi rh$

$$= 2\times\pi\times 12\times 10 \text{ m}^2$$
$$= 240\pi \text{ m}^2$$

Radius of the cone $= 12$ m

Height of the cone $= 5$ m

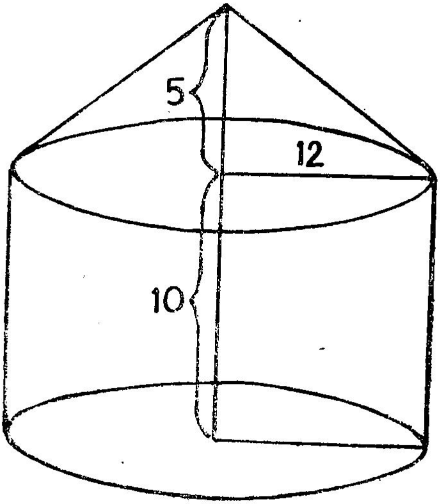


Fig. 5-15

$$l = \sqrt{h^2 + r^2}$$
$$= \sqrt{5^2 + 12^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13 \text{ m}$$

Slant height $= 13$ m

Curved surface area of the cone $= \pi r l$

$= \pi \times 12 \times 13 \text{ m}^2$

$= 156\pi \text{ m}^2$

Area of the canvas required $= 240\pi + 156\pi$

$= 396\pi \text{ m}^2$

Exercise 5·2·7

1. The lower part of a tent is in the form of a cylinder of diameter 40 m and height 3·5 m. The upper part of the tent is a cone. The total height of the tent from the ground is 18·5 m. Find the area of the canvas required to make the tent.

2. Scented oil is filled in a cylindrical vessel whose radius is 2·5 cm and height 8 cm. This oil is poured into conical cups. The radius and height of the conical cups are 1 cm and 5 cm respectively. Find the number of conical cups required.

3. The ratio of the radii of the bases of a cone and a cylinder is 3 : 5. The ratio of their heights is 2 : 3. What is the ratio of their volumes ?

4. The radius and height of a cone are 2 cm and 5 cm respectively. 15 such cones are dropped into a cylindrical jar of 10 cm radius, partly filled with water. How much will the level of water rise in the cylinder ?

5. The diameters of two cones are 6 cm and 8 cm. They are of equal height. They are melted and cast into a cylinder with the same height. What is the radius of the cylinder thus made ?

ANSWERS

(1) $640\pi m^2$ (2) 30 (3) 2 : 25 (4) 1 cm
(5) 2·88 cm

5-2-8. Cylinder and Sphere

Example 1 :

The upper part of a water tank is in the form of a cylinder and the lower part is a hemisphere. The common diameter is 3 dm. Its depth is 13 dm. Find its capacity.

Solution :

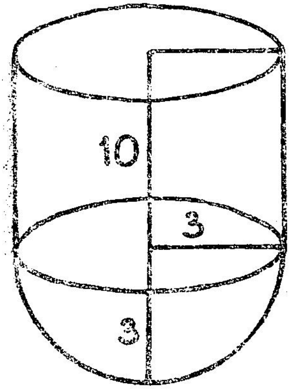


Fig. 5-16

Capacity of the water tank = Volume of the cylindrical Part + Volume of the hemispherical part

$= \pi r^2 h + \frac{2}{3} \pi r^3$

$= \pi r^2 \left(h + \frac{2}{3} r \right)$

$= \pi \times 3^2 \times \left(10 + \frac{2}{3} \times 3 \right)$ dm^3

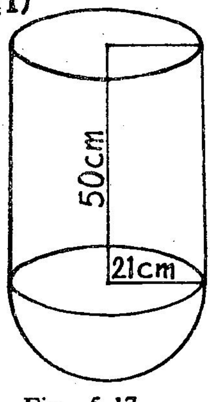
$= \pi \times 9 \times 12$ dm^3

$= 108\pi$ dm^3

Exercise 5-2·8

1. Find the volume of the following solids.

(I)



(ii) Fig. 5-17.

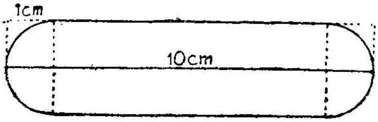


Fig. 5-18.

Total length = 10 cm.

2. A cylinder whose radius is 14 cm and height 1 m is melted and cast into spheres of 1 cm diameter each. How many spheres can be made ?

ANSWERS

(1) (a) 28224π cm³ (b) $11\frac{1}{3}\pi$ cm³

(2) 117600

5-2.9 Cone and Sphere

Example :

The lower part of a solid is in the form of a hemisphere and its upper part in the form of a cone. The height and radius of the cone are 12 cm and 4 cm respectively. Find the volume of the solid.

Solution :

Volume of the cone

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \pi \times 4 \times 4 \times 12 \text{ cm}^3$$

$$= 64\pi \text{ cm}^3$$

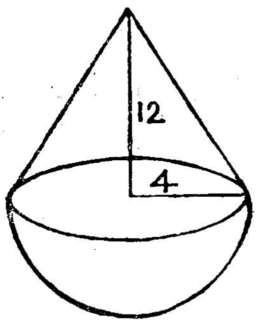


Fig. 5-19.

Volume of the hemisphere

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times \pi \times 4 \times 4 \times 4 \text{ cm}^3$$

$$= \frac{128}{3}\pi \text{ cm}^3$$

$$= 42\frac{2}{3}\pi \text{ cm}^3$$

Volume of the solid

$$= \left(64\pi + 42\frac{2}{3}\pi\right) \text{cm}^3$$

$$= 106\frac{2}{3}\pi \text{ cm}^3$$

Exercise 5-2·9

1. Find the volume of the following solid.

(i)

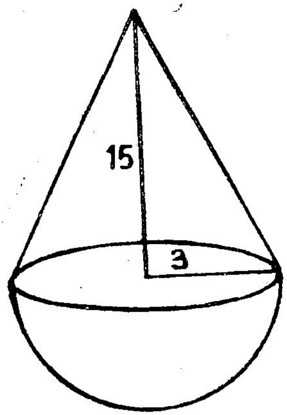


Fig. 5-20

(ii)

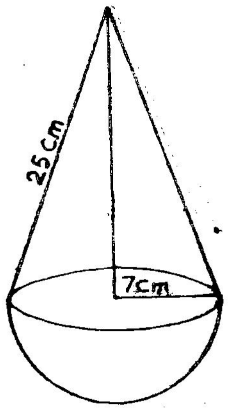


Fig. 5-21.

2. A plastic doll has the shape of a hemisphere surmounted by a cone. The radius of the hemisphere is 5 cm. The slant height of the cone is 11 cm. What is the area of the plastic sheet required to make the doll ?

3. A metal cone of radius 20 cm and height 30 cm is melted and cast into spheres of radius 2 cm each. How many spheres can be made out of it ?

ANSWERS

(1) $620\frac{2}{3}\pi$ cm^3 (2) 105π cm^2

5·3 Logarithms

5·3·1 Logarithms : Definition

$a^x = b$ ($a \neq 0$, $a \neq 1$, $a > 0$, $b > 0$, $a, b, x \in R$) gives a relation between a, b and x. We know that a is the base and x is the exponent. We also know that $\sqrt[x]{b} = a$.

$$\text{Since } 5^2 = 25,\ 5 = \sqrt{25}.$$

If we find the value of x in terms of a and b, the solution is called the **logarithm** of b to the **base** a. It is written as

$$x = \log_a b$$

Note that a is also the base for the exponent.

$a^x = b$ is the **exponential equation.**

Using logarithm this equation can be written as

$$x = \log_a b$$

Both these equations tell us the same relation between a, b and x in different ways.

Equation	Logarithmic Equation
$2^3 = 8$	$\log_2 8 = 3$
$3^4 = 81$	$\log_3 81 = 4$
$2^{-3} = \frac{1}{8}$	$\log_2 \frac{1}{8} = -3$
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$

Note :

(1) $\log 1 = 0$ (to any base)

(2) log of a number to the same base is 1.

Example 1 :

Find the value of $\log_2 64$.

Solution :

$$\text{Let } \log_2 64 = a$$
$$2^a = 64$$
$$2^a = 2\times2\times2\times2\times2\times2 = 2^6$$
$$\therefore \quad a = 6$$
$$\log_2 64 = 6$$

Example 2 :

If $\log_7 x = 3$, find x.

Solution :

$$\log_7 x = 3$$
$$x = 7^3$$
$$= 7 \times 7 \times 7$$
$$x = 243$$

Exercise 5-3·1

1. Rewrite the following equations using exponents.
   (a) $\log_3 9 = 2$ (b) $\log_6 216 = 3$
   (c) $\log_5 625 = 4$ (d) $\log_2 \frac{1}{16} = -4$
   (e) $\log_a m = n$ (f) $\log_x P = q$

2. Rewrite the following equations using logarithms.
   (a) $2^5 = 32$ (b) $3^6 = 729$ (c) $125^{\frac{1}{3}} = 5$
   (d) $\frac{1}{81} = 3^{-4}$ (e) $m^n = l$ (f) $k^l = a$

3. Find the value of x in the following.
   (a) $\log_2 256 = x$ (b) $\log_5 3125 = x$
   (c) $\log_4 \frac{1}{64} = x$ (d) $\log_4 x = 5$
   (e) $\log_{\frac{1}{2}} x = 3$

4. Find the value of :

(a) $\log_{10} 1000$ (b) $\log_8 16$

(c) $\log_9 3$ (d) $\log_{10} \cdot 001$

ANSWERS

(1) (a) $3^2 = 9$ (2) (a) $\log_2 32 = 5$

(3) (a) $x = 8$ (4) (a) 3

5-3·2 Logarithmic Rules

Recall : $a^x \times a^y = a^{x+y}$; $a^x \div a^y = a^{x-y}$

1. Let $\log_a m = x$, $\log_a n = y$

$$m = a^x,\ n = a^y$$
$$mn = a^x \times a^y$$
$$= a^{x+y}$$
$$\log_a (mn) = x + y$$
$$= \log_a m + \log_a n$$

Note :

$$\log_a m + \log_a n \neq \log_a (m + n) \text{ in general.}$$

2. Let $\log_a m = x$, $\log_a n = y$

$$m = a^x,\ n = a^y$$
$$\frac{m}{n} = \frac{a^x}{a^y}$$
$$= a^{x-y}$$
$$\log_a \left(\frac{m}{n}\right) = x - y$$
$$= \log_a m - \log_a n$$

3 Let $\log_a m = x$

$$m = a^x$$
$$m^n = (a^x)^n$$
$$= a^{nx}$$
$$\log_a m^n = nx$$
$$= n \log_a m$$

Remember :

$$\log_a (mn) = \log_a m + \log_a n$$
$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$
$$\log_a m^n = n \log_a m$$

Example 1 :

Verify $\log_{10} 1600 = 2 + 4 \log_{10} 2$

Solution :

$$\begin{aligned}\log_{10} 1600 &= \log_{10} (100 \times 16)\\ &= \log_{10} 100 + \log_{10} 16\\ &= 2 + \log_{10} 2^4\\ &= 2 + 4\log_{10} 2\end{aligned}$$

Example 2 :

Prove ; $\log_{10} \left(11 \frac{1}{9}\right) = 2 - 2 \log_{10} 3$

Solution :

$$\begin{aligned}\log_{10} \left(11 \frac{1}{9}\right) &= \log_{10} \left(\frac{100}{9}\right)\\ &= \log_{10} 100 - \log_{10} 9\\ &= 2 - \log_{10} 3^2\\ &= 2 - 2\log_{10} 3\end{aligned}$$

Example 3 :

Prove : $\log_{10} \frac{15}{16} + \log_{10} \frac{64}{81} - \log_{10} \frac{4}{27} = \log_{10} 5$

Solution :

$$\log_{10} \frac{15}{6} + \log_{10} \frac{64}{81} - \log_{10} \frac{4}{27}$$
$$= \log_{10} \left(\frac{15}{16} \times \frac{64}{81} \div \frac{4}{27}\right)$$
$$= \log_{10} \left(\frac{15}{16} \times \frac{64}{81} \times \frac{27}{4}\right)$$
$$= \log_{10} 5$$

Example 4 :

log 3 = 0·4771, log 5 = 0·6990, log 7=0·8451. Find log 105.

Solution :

$$105 = 3 \times 5 \times 7$$

$$\begin{aligned} \log 105 &= \log (3 \times 5 \times 7) \\ &= \log 3 + \log 5 + \log 7 \\ &= 0{\cdot}4771 + 0{\cdot}6990 + 0{\cdot}8451 \\ &= 2{\cdot}0212 \end{aligned}$$

Exercise 5-3-2

1. Simplify :
   - (a) $\log_{10}25 + \log_{10}4$
   - (b) $\log_{12}16 + \log_{12}9$
   - (c) $\log_{10}15 - \log_{10}5$
   - (d) $\log_{10}64 - \log_{10}16$
   - (e) $\log_{10}12 + \log_{10}20 + \log_{10}15$
   - (f) $\log_{10}30 + \log_{10}20 - \log_{10}10$

2. If log 2 = 0·3010, log 3 = 0·4771, log 5 = 0·6990, log 7 = 0·8451. Find the value of the following.

   (1) log 15 (2) log 6 (3) log 35 (4) log 21 (5) $\log 7^2$

   (6) $\log 5^4$ (7) $\log 2^8$

3. Prove : $\log_{10}72 = 2\log_{10}3 + 3\log_{10}2$

4. Prove : $\log_{10}2500 = 2 + 2\log_{10}5$

5. Prove : $\log_{10}\left(2\frac{2}{49}\right) = 2 - 2\log_{10}7$

6. Prove : $\log_{10}125 = 3 - 3\log_{10}2$

7. Prove : $\log_{10}\frac{9}{8} - \log_{10}\frac{27}{32} + \log_{10}\frac{3}{4} = 0$

8. Simplify : $\log_5\frac{5}{12} - \log_5\frac{7}{5} - \log_5\frac{7}{12}$

9. Simplify : $\log_{10}\frac{26}{33} - \log_{10}\frac{65}{69} + \log_{10}\frac{55}{46}$

ANSWERS

1. (a) 2 2 (1) 1·1761 (8) $\log_5 \frac{25}{49}$ (9) 0

5-3.3. Logarithmic tables : Construction

NOT FOR EXAMINATION
MATHEMATICS CLUB — ACTIVITY

$2^{10} = 1024 \simeq 1000 = 10^3$

$2 = (2^{10})^{\frac{1}{10}} \simeq (10^3)^{\frac{1}{10}} = 10^{\cdot 3}$

$\therefore \log 10^2 \simeq \cdot 3$

$\log_{10}4 = \log_{10}2^2 = 2\log_{10}2 \simeq \cdot 6$

$\log_{10}8 = \log_{10}2^3 = 3\log_{10}2 \simeq \cdot 9$

$\log_{10}5 = \log_{10}\frac{10}{2} = \log_{10}10 - \log_{10}2$

$\simeq 1 - \cdot 3 = \cdot 7$

$3^4 = 81 \simeq 80 = 8 \times 10 = 10^{\cdot 9} \times 10^1 = 10^{1 \cdot 9}$

$3 = (10^{1\cdot 9})^{\frac{1}{4}} = 10^{0\cdot 475}$

$\therefore \log_{10}{}^3 \simeq 0\ 475$

$\log_{10}{}^9 = \log_{10}3^2 \simeq 2\log 10^3$

$= 2 \times \cdot 475$

$= \cdot 95$

$\log_{10}6 = \log_{10}(2\times 3) = \log_{10}2 + \log_{10}3$

$\simeq \cdot 3 + \cdot 475 = \cdot 775$

$7^2 = 49 \simeq 50 = 5 \times 10$

$= 10^{\cdot 7} \times 10^1 = 10^{1\cdot 7}$

$7 \simeq (10^{1\cdot 7})^{\frac{1}{2}} = 10^{\ 85}$

$\therefore \log 7 \simeq \cdot 85$

No.	1	2	3	4	5	6	7	8	9	10
log	0	3	·475	·6	·7	·775	·85	·9	·95	1

NOTE

1. $\log_{10}2 = 0{\cdot}3$

$$\begin{aligned}\log_{10}20 &= \log_{10}(10\times 2)\\ &= \log_{10}10 + \log_{10}2\\ &= 1 + 0{\cdot}3\\ &= 1{\cdot}3\end{aligned}$$

$$\begin{aligned}\log_{10}200 &= \log_{10}(10\times 10\times 2)\\ &= \log_{10}10 + \log_{10}10 + \log_{10}2\\ &= 1 + 1 + 0{\cdot}3\\ &= 2{\cdot}3\end{aligned}$$

$$\begin{aligned}\log_{10}2000 &= \log_{10}(10\times 10\times 10\times 2)\\ &= \log_{10}10 + \log_{10}10 + \log_{10}10 + \log_{10}2\\ &= 1 + 1 + 1 + 0{\cdot}3\\ &= 3{\cdot}3\end{aligned}$$

2. $\log_{10}2 = 0{\cdot}3$
$\log_{10}20 = 1{\cdot}3$
$\log_{10}200 = 2{\cdot}3$
$\log_{10}2000 = 3{\cdot}3$

The whole number part of the logarithm of any number is known as the **characteristic** and the decimal part is known as the **mantissa**.

We commonly use logarithms to the base 10. So unless stated otherwise, the base of the logarithm of any number will be taken as 10.

3

Given number	Number of digits	Characteristic	Mantissa	Logarithm
3	1	0	·475	0·475
30	2	1	·475	1 475
300	3	2	475	2·475
3000	4	3	·475	3·475

4.

Number	Characteristic	Mantissa	Logarithm
4000	3	·6	3·6
400	2	·6	2·6
40	1	·6	1·6
4	0	·6	0·6
·4	−1 or $\bar{1}$	·6	$\bar{1}$·6
·04	−2 or $\bar{2}$	·6	$\bar{2}$·6
·004	−3 or $\bar{3}$	·6	$\bar{3}$·6

5·3·4 Logarithmic Tables : **Use of log and antilog tables**

Example 1 :

Find log 3476 using tables.

Solution :

The number of digits in **3476 is 4.**

Therfore, the characteristic of **3476** is 3.

From the log table :

Logarithms

		7		Mean Differences		
					6	
3·4		·5403			8	

$$\log \text{ of } 3{\cdot}476 = 0{\cdot}5403 + 0{\cdot}0008$$
$$= 0{\cdot}5411$$
$$\therefore \quad \log \text{ of } 3476 = 3{\cdot}5411$$

Example 2 :

The log of a **number** is **2·7312. Find the number.**

Solution :

From the antilog table :

Anti Logarithms

	1		Mean Differences		
				2	
·73	5 383			3	

antilog of ·7312 = 5·383 + 0·003

= 5·386

The characteristic in log 2·7312 is 2.

Therefore, the number of digits in the required number is 3.

Antilog of 2 7312 is 538·6.

The required number is 538·6.

Exercise 5-3·4

1. Find logarithms for the following.

   (1) 4137 (2) 7234 (3) 648·5 (4) 72·15 (5) 4·132

   (6) ·5413 (7) 43612 (8) 2·64

2. The logarithms of certain numbers are given below. Find the numbers

   (1) 2·514 (2) 3·712 (3) 4·222 (4) 1·234 (5) ·5174

5·3·5 Logarithm Tables: Use For Simplification

Note the logarithm operations equivalent to number operations.

Number Operation	Log Operation	Example
Multiplication	Addition	$\log (152 \times 324) = \log 152 + \log 324$
Division	Subtraction	$\log (400 \div 34) = \log 400 - \log 34$
Exponential	Multiplication	$\log 23^5 = 5 \log 23$
Root	Division	$\log \sqrt{30} = \frac{1}{2} \log 30$

Example 1 :

Simplify using tables :

434·38 × 37·48

Solution :

No. Operation	Numbers	Log	Log operations
×	434·38	2·6378	+
	37·48	1·5738	
		4·2116	

antilog 4·2116 = 16280

Example 2 :

Simplify using tables.

$$\frac{4{\cdot}325}{0{\cdot}031}$$

Solution :

No. Operation	Numbers	Log	Log operation
÷	4·325	0·6360	−
	·031	−2·4914	
		2·1446	

antilog 2·1446 = 139 5

Note : $-2{\cdot}4914 = -2 + {\cdot}4914$

$$0{\cdot}6360 - (-2{\cdot}4914) = 0{\cdot}6360 - (-2 + {\cdot}4914)$$
$$= 0{\cdot}6360 + 2 - 0{\cdot}4914$$
$$= 2{\cdot}1446$$

Example 3 :

Using tables, find the value of $(41{\cdot}37)^5$

Solution :

No. Operation	Number	Log	Log operation
Exponential	41·37	1 6167	×
	$(41{\cdot}37)^5$	5 × 1 6167	
		= 8·0835	

antilog 8·0835 = 121200000

Example 4 :

Using log tables find the square root of 561·7

Solution :

No. Operation	Numbers	Log	Log Operation
Square root	561·7 $\sqrt{(561\cdot7)}$	2·7495 $\frac{1}{2} \times 2\cdot7495$ $= 1\cdot3747$	÷

antilog 1·3747 = 23·70

Example 5 :

Using log tables, find the cube root of 75·13.

Solution :

No. Operation	Numbers	Log	Log Operation
Cube root	75·13 $\sqrt[3]{(75\cdot13)}$	1·8758 $\frac{1}{3} \times 1\cdot8758$ $= 0\cdot6252$	÷

antilog 0·6252 = 4·219

Exercise 5-3-5

Simplify the following using logarithmic tables.

(1) 4132×5268 (2) $723\cdot5 \times 61\cdot56$

(3) $4\cdot2576 \times \cdot00271$ (4) $3\cdot713 \div 2\cdot52$

(5) $473\cdot4 \div 1\cdot635$ (6) $0\cdot0123 \div 4\cdot123$

(7) $(54\cdot67)^7$ (8) $(33\cdot47)^6$

(9) $\sqrt{431\cdot4}$ (10) $\sqrt[3]{7229}$

ANSWERS

(1) 21770000 (4) 1·473

(7) 1461000000000 (9) 20·77

REVIEW

1. Solids :

Name of solid	Lateral or curved Surface Area	Total surface Area	Volume
1. Prism P = Perimeter of base A = Area of base h = Height	Ph	$Ph + 2A$	Ah
2. Cylinder r = Radius h = Height	$2\pi rh$	$2\pi r(h + r)$	$\pi r^2 h$
3. Cone r = Radius h = Height l = Slant height $l^2 = h^2 + r^2$	πrl	$\pi r(l + r)$	$\frac{1}{3}\pi r^2 h$
4 Sphere r = radius	$4\pi r^2$	—	$\frac{4}{3}\pi r^3$
5. Hemisphere	$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$

2. A sector is folded into a cone.

The slant height of the cone = radius of the sector.

$$\text{Radius of the cone} = \text{radius of the sector} \times \frac{\text{central angle}}{360°}$$

3 The relation $a^x = b$ can be written as

$$x = \log_a b \text{ in log form.}$$

$$\log mn) = \log m + \log n$$

$$\log \frac{m}{n} = \log m - \log n$$

$$\log m^n = n \log m$$

No. operation	log operation	Example
Multiplication	Addition	$\log (15 \times 48) = \log 15 + \log 48$
Division	Subtraction	$\log \left(\frac{524}{37}\right) = \log 534 - \log 37$
Exponential	Multiplication	$\log 8^5 = 5 \log 8$
Root	Division	$\log \sqrt[4]{58} = \frac{1}{4} \log 58$

SELF-ASSESSMENT TEST

1. The height of a regular hexagonal prism of side 7 cm is 12 cm. Calculate the total surface area and the volume of the prism

2. The radius and height of a cylinder are 3 cm and 10 cm respectively. Find its curved surface area and volume.

3. The slant height of a cone is 39 cm. Its radius is 15 cm. Find its volume and total surface area.

4. The volume of a sphere is 36π cm^3. Calculate its surface area.

5. A hard metal cylinder of radius 20 cm and height 160 cm is melted and made into spheres of radius 4 cm. Find the number of spheres made.

6. Show that $\log_{10} 64000 = 3 + 6 \log_{10} 2$

7. Simplify using log tables.

(a) 4379×1234 (b) $2439 \div 1{\cdot}37$ (c) $(32{\cdot}64)^5$

(d) $\sqrt{92167}$ (e) $\sqrt[3]{7455}$

NOT FOR EXAMINATION

MATHEMATICS CLUB—ACTIVITY 5

In VIII Standard we have seen the five Platonic solids named according to thier shapes.

Let us now examine one of their properties. We can make such models with tubes (eg. straw) and wires (eg. pipe cleaners).

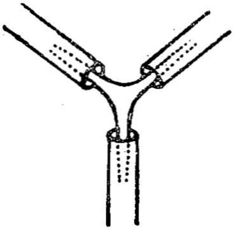
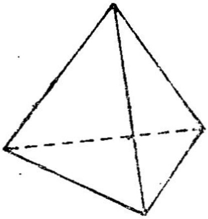
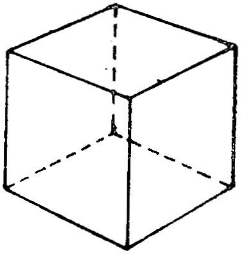


Fig. 5·22



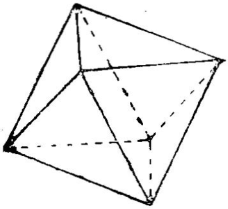
TETRAHEDRON

Fig. 5·23



HEXAHEDRON
(Cube)

Fig. 5·24



OCTAHEDRON

Fig. 5·25

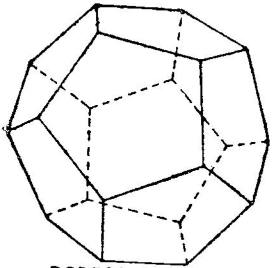


Fig. 5·26

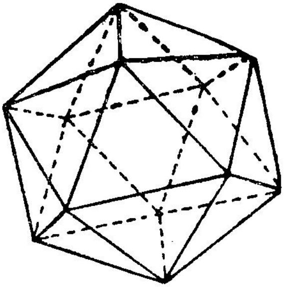


Fig. 5·27

Find the number of faces, the number of edges and the number of vertices in each of the five solids and tabulate them as shown below. Learn the relation that exists among them.

Regular solid	faces (f)	edges (e)	vertices (v)
Tetrahedron	6	12	8
Hexahedron (cube)			
Octahedron			
Dodecahedron			
Icosahedron			

Tabulate the above particulars for different prisms. Find whether the relation $v + f = e + 2$ found above holds good for them also.

This relation was discovered by Euler in 1752.

6. MATRICES

1. Matrices : Definition

The marks secured by a few students in a particular test are given below.

Student	Tamil	English	Maths	Science	His. & Geography
A	40	50	60	42	47
B	60	65	68	70	53
C	25	15	30	12	40

This may be written as

$$\begin{pmatrix} 40 & 50 & 60 & 42 & 47 \\ 60 & 65 & 68 & 70 & 53 \\ 25 & 15 & 30 & 12 & 40 \end{pmatrix}$$

where the rows denote the marks of the students and the colums denote the subjects. This rectangular arrangement of information is called a **Matrix** and helps to present in a concise form numerous data.

The order of a matrix is denoted by the number of its rows and columns. There are 3 rows and 5 columns in the above matrix. The matrix is therefore said to be of order 3×5.

A matrix of order 4×3 has 4 rows and 3 columns. The individual numbers in a matrix are called its **elements.** There are 15 elements in the above matrix.

How many elements will there be in a matrix of order 4×3?

Row Matrix

A matrix with a single row is called **a row matrix** or **a row vector**.

e. g; (1 3 5 7) is a row matrix of order 1 × 4.

Column Matrix

A matrix with a **single column** is called a **column matrix** or a **column vector.**

e.g; $\begin{pmatrix} 4 \\ 7 \\ 1 \\ 3 \end{pmatrix}$ is a column matrix its order in 4×1.

Square Matrix

A matrix with **equal** number of rows and column is called a **square matrix.**

e.g , $\begin{pmatrix} 4 & 7 \\ 3 & 2 \end{pmatrix}$ is a square matrix of order 2×2.

Zero Matrix

A matrix in which every element is zero is called a **Zero matrix.** Thus $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ are zero matrices.

Equal Matrices

Two matrices are said to be **equal matrices** when they are of the same order and have equal entries, position-wise.

Their corresponding elements are equal.

eg. $\begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 9-6 & \sqrt{16} \\ 1+1 & \frac{14}{2} \end{pmatrix}$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then

$a = e$, $b = f$, $c = g$ and $d = h$.

Exercise 6-1

1. State the order of the following matrices. How many elements are there in each matrix ?

(a) $\begin{pmatrix} 5 & 3 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 5 \\ 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 2 \\ 7 & 1 \\ 5 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 7 & 5 \\ 5 & 8 & 6 \\ 7 & 1 & 1 \\ 4 & 2 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 5 & 1 & 2 & 3 \\ 4 & 6 & 8 & 1 \\ 2 & 1 & 3 & 5 \end{pmatrix}$

2. Form the zero matrix of the following orders.
3×2, 2×2, 3×3

3. If $(a \quad 4) = (3 \quad b)$, find a and b.

4. If $\begin{pmatrix} x & 5 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 0 & m \\ n & 8 \end{pmatrix}$, find the values of x, y, m and n.

5. If $\begin{pmatrix} x - 11 \\ y + 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, find x and y.

ANSWERS

(1) (a) 1×3 (b) 2×2, (c) 3×2, (d) 4×3
(2) 4×4, (3) $a = 3$, $b=4$ (4) $x=0$, $m=5$, $n=3$, $y=8$
(5) $x = 16$, $y = -4$.

6·2 Matrices : Addition

1. Particulars of provisions bought by two persons during two months are give below.

(i) **January 1982**

	Rice	Dhall	Oil
Anbu	15 kg	4 kg	3 litres
Anandam	10 kg	6 kg	4 litres

(ii) February 1982

	Rice	Dhall	Oil
Anbu	10 kg	4 kg	4 litres
Anandam	12 kg	0 kg	5 litres

The total purchase in the above two months will be

	Rice	Dhall	Oil
Anbu	(15 + 10) kg	(4 + 4) kg	(3 + 4) litres
Anandam	(10 + 12) kg	(6 + 0) kg	(4 + 5) litres

We can record the same information in matrix form.

$$\begin{pmatrix} 15 & 4 & 3 \\ 10 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 10 & 4 & 4 \\ 12 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 15+10 & 4+4 & 3+4 \\ 10+12 & 6+0 & 4+5 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 8 & 7 \\ 22 & 6 & 9 \end{pmatrix}$$

If the quantity of oil bought in February is not known, can we find the total purchase of oil for both the months? No, we cannot.

$$\begin{pmatrix} 15 & 4 & 3 \\ 10 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 10 & 4 & ? \\ 12 & 6 & ? \end{pmatrix} = \begin{pmatrix} 25 & 8 & 3+? \\ 22 & 6 & 4+? \end{pmatrix}$$

So, we can add matrices of the same order only.

For matrices of different orders, addition is not defined. In addition, the corresponding entries are added. The sum of two matrices of the same order is also a matrix of the same order.

Example 1 :

Find $\begin{pmatrix} 4 & 3 & 5 \\ 2 & 6 & 8 \end{pmatrix} + \begin{pmatrix} 7 & 1 & 0 \\ 5 & 6 & 3 \end{pmatrix}$

Solution :

$$\begin{pmatrix} 4 & 3 & 5 \\ 2 & 6 & 8 \end{pmatrix} + \begin{pmatrix} 7 & 1 & 0 \\ 5 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4+7 & 3+1 & 5+0 \\ 2+5 & 6+6 & 8+3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 4 & 5 \\ 7 & 12 & 11 \end{pmatrix}$$

In general,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} + \begin{pmatrix} p & q & r \\ m & n & l \\ x & y & z \end{pmatrix}$$

$$= \begin{pmatrix} a+p & b+q & c+r \\ d+m & e+n & f+l \\ g+x & h+y & k+z \end{pmatrix}$$

Example 2 :

Find $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 7 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix}$

Solution :

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 7 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 2+6+5 & 3+7+4 \\ 4+1+2 & 5+3+9 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 14 \\ 7 & 17 \end{pmatrix}$$

Exercise 6-2

Add the following matrices, if possible. If it is not possible, state the reason.

1. $(3 \quad 5) + (7 \quad 4)$
2. $(2 \quad 7) + (4 \quad -3)$
3. $(4 \quad 6 \quad 7) + (11 \quad 5 \quad -3)$

4. $(5\ \ 9\ \ 8\ \ 7) + (3\ \ 1\ \ 3)$

5. $\begin{pmatrix} 7 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

6. $(2\ \ 3\ \ -1) + (5\ \ 6\ \ 2) + (4\ \ 7\ \ -3)$

7. $\begin{pmatrix} 9 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

8. $\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 7 & -5 \\ 6 & -4 \end{pmatrix} + \begin{pmatrix} -5 & 6 \\ 7 & 2 \end{pmatrix}$

10. $\begin{pmatrix} 4 & 4 & -2 \\ 5 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 7 & 2 & -1 \\ 8 & 5 & 6 \end{pmatrix}$

11. $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + (2\ \ 3\ \ 5)$

12. $\begin{pmatrix} 1 & 4 & 5 \\ 2 & -1 & 7 \\ 5 & 1 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 6 \\ 7 & 1 & 8 \\ 6 & -5 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -9 & 2 \\ 5 & 11 & 8 \\ 4 & 7 & 0 \end{pmatrix}$

13. If $A = \begin{pmatrix} 5 & 7 \\ 9 & -1 \\ 4 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -3 \\ 7 & 5 \\ 4 & 6 \end{pmatrix}$ $C = \begin{pmatrix} 1 & -1 \\ -2 & 3 \\ 5 & 6 \end{pmatrix}$

Find (a) $A + B$ (b) $B + C$ (c) $C + A$.

ANSWERS

(1) $(10\ \ 9)$ (5) $\begin{pmatrix} 11 \\ 13 \\ 10 \end{pmatrix}$ (9) $\begin{pmatrix} 2 & 1 \\ 13 & -2 \end{pmatrix}$

(12) $\begin{pmatrix} 9 & -1 & 13 \\ 11 & 11 & 23 \\ 15 & 3 & 8 \end{pmatrix}$ (4), (7) and (11) cannot be added.

6·3 Matrices : Subtraction

Negative of Matrix

Given a matrix $A = \begin{pmatrix} 1 & -2 & -5 \\ 1 & -4 & 6 \\ 7 & 1 & 3 \end{pmatrix}$,

we can construct a matrix B, by replacing each element in A by its additive inverse.

$$B = \begin{pmatrix} -1 & 2 & 5 \\ -1 & 4 & -6 \\ -7 & -1 & -3 \end{pmatrix}$$

Find $A + B$ and $B + A$.

Note that $A + B = B + A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$

B is called the **negative matrix** of A. Note that the order of B and A are the same and the elements of B are the additive inverses of the corresponding entries of the first matrix A. We denote the negative matrix of A by $-A$.

Example :

The negative matrix of

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & -4 & -2 \end{pmatrix} \text{ is } \begin{pmatrix} -2 & 3 & -4 \\ -3 & 4 & 2 \end{pmatrix}$$

Subtraction

If A and B are two matrices of the same order, we define $A - B$ as $A + (-B)$.

Example 1 :

If $A = \begin{pmatrix} -3 & 4 \\ 2 & -7 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -5 \\ 6 & -3 \end{pmatrix}$, find $A - B$.

Solution :

We can find $A - B$ by subtracting the entries of the second matrix B from the corresponding entries of the first matrix A

$$A - B = \begin{pmatrix} -3 & 4 \\ 2 & -7 \end{pmatrix} - \begin{pmatrix} -1 & -5 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4 \\ 2 & -7 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3+1 & 4+5 \\ 2-6 & -7+3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 9 \\ -4 & -4 \end{pmatrix}$$

Example 2 :

Simplify :

$$\begin{pmatrix} 3 & 4 & -5 \\ -2 & 3 & -4 \\ 5 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & 4 \\ 2 & -4 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 2 & -3 \\ 3 & -5 & 2 \\ -2 & 3 & -5 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} 3 & 4 & -5 \\ -2 & 3 & -4 \\ 5 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & 4 \\ 2 & -4 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 2 & -3 \\ 3 & -5 & 2 \\ -2 & 3 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & -5 \\ -2 & 3 & -4 \\ 5 & -2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 2 \\ -2 & -3 & -4 \\ -2 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 5 & 2 & -3 \\ 3 & -5 & 2 \\ -2 & 3 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3-1+5 & 4+0+2 & -5+2-3 \\ -2-2+3 & 3-3-5 & -4-4+2 \\ 5-2-2 & -2+4+3 & 3-3-5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 6 & -6 \\ -1 & -5 & -6 \\ 1 & 5 & -5 \end{pmatrix}$$

Exercise 6·3

1. Find the negative matrices of the following.

(a) $\begin{pmatrix} -5 & 4 \\ 3 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -3 & 5 \\ -2 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 5 & 1 \\ 2 & -1 & 3 \\ -4 & -3 & -7 \end{pmatrix}$

2. Find the negative of a zero matrix.

Simplify :

3. $(1 \quad 5) - (2 \quad -7)$

4. $(-1 \quad 3 \quad 4) - (-5 \quad 6 \quad -2)$

5. $\begin{pmatrix} 1 & 5 \\ -7 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 5 & -9 \end{pmatrix}$

6. $\begin{pmatrix} -1 & 4 & 7 \\ 5 & -6 & 3 \end{pmatrix} - \begin{pmatrix} -5 & 2 & 6 \\ 4 & -7 & -1 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 4 & 7 \\ -5 & -6 & 3 \end{pmatrix} - \begin{pmatrix} -5 & 2 & -6 \\ 4 & -7 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ -5 & 7 \end{pmatrix} - \begin{pmatrix} -3 & -2 \\ 1 & 7 \end{pmatrix}$

9. $\begin{pmatrix} 1 & -3 & 5 \\ 2 & 7 & 5 \end{pmatrix} - \begin{pmatrix} 4 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & -7 \end{pmatrix}$

10. If $A = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -4 & 6 \\ 7 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & -3 \\ 5 & 4 \end{pmatrix}$

Find (a) $A + B + C$ (b) $A - B + C$ (c) $A - B - C$
(d) $A + B - C$ (e) $C - A + B$

ANSWERS

1. (a) $\begin{pmatrix} 5 & -4 \\ -3 & 7 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & -5 & -1 \\ -2 & 1 & -3 \\ 4 & 3 & 7 \end{pmatrix}$

3. $(-1 \quad 12)$ 5. $\begin{pmatrix} -1 & 2 \\ -12 & 8 \end{pmatrix}$ 7. $\begin{pmatrix} 6 & 2 & 13 \\ -9 & 1 & 2 \end{pmatrix}$

9. $\begin{pmatrix} -2 & -1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$ 10. (b) $\begin{pmatrix} 3 & -4 \\ 1 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} -9 & -2 \\ 9 & 8 \end{pmatrix}$

6·4 Matrix Equations

Recall

In real numbers 0 is the additive identity.

$$4 + 0 = 4$$

Similarly zero matrix is the additive identity in matrices.

e. g. $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4+0 & 5+0 \\ 2+0 & 3+0 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

If we are given two matrices A and B of the same order we can find the matrix X such that $X + A = B$.

$$X + A = B$$
$$X + A + (-A) = B + (-A)$$
$$X + 0 = B - A$$
$$X = B - A$$

Example 1 :

Solve :

$$\begin{pmatrix} 2 & -5 \\ 3 & 4 \end{pmatrix} + X = \begin{pmatrix} -4 & 2 \\ 1 & -3 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} 2 & -5 \\ 3 & 4 \end{pmatrix} + X = \begin{pmatrix} -4 & 2 \\ 1 & -3 \end{pmatrix}$$

$$X = \begin{pmatrix} -4 & 2 \\ 1 & -3 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 -2 & 2 + 5 \\ 1 -3 & -3 - 4 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 7 \\ -2 & -7 \end{pmatrix}$$

Example 2 :

Solve :

$$\begin{pmatrix} 1 & -3 & 2 \\ 5 & 1 & -4 \end{pmatrix} - X = \begin{pmatrix} -2 & -1 & 3 \\ 4 & 5 & 2 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} 1 & -3 & 2 \\ 5 & 1 & -4 \end{pmatrix} - X = \begin{pmatrix} -2 & -1 & 3 \\ 4 & 5 & 2 \end{pmatrix}$$

$$-X = \begin{pmatrix} -2 & -1 & 3 \\ 4 & 5 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -3 & 2 \\ 5 & 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 3 \\ 4 & 5 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 & -2 \\ -5 & -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & 1 \\ -1 & 4 & 6 \end{pmatrix}$$

$$X = \begin{pmatrix} 3 & -2 & -1 \\ 1 & -4 & -6 \end{pmatrix}$$

Exercise 6·4

Solve :

1. $X + (1 \quad 2) = (5 \quad 7)$
2. $(3 \quad -4) - X = (-1 \quad 2)$
3. $X - (4 \quad 3 \quad -7) = (1 \quad 5 \quad -6)$
4. $X + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 3 \end{pmatrix}$

5. $X + \begin{pmatrix} 2 & 3 \\ -5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -9 \\ -2 & 8 \end{pmatrix}$

6. $\begin{pmatrix} 4 & -1 \\ 5 & 2 \end{pmatrix} - X = \begin{pmatrix} -5 & 6 \\ 0 & -3 \end{pmatrix}$

7. $X + \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 \\ -4 & 1 & 7 \end{pmatrix}$

8. Given $A = \begin{pmatrix} 4 & 5 \\ 2 & -3 \\ -4 & 5 \end{pmatrix}$ find matrix X such that $X + A = 0$

ANSWERS

(1) $(4 \quad 5)$ (3) $(5 \quad 8 \quad -13)$ (5) $\begin{pmatrix} -1 & -12 \\ 3 & 1 \end{pmatrix}$

(7) $\begin{pmatrix} -2 & 1 & 1 \\ -3 & -4 & 10 \end{pmatrix}$

6·5 Matrices : Scalar Multiplication

If $A \in R$, we define $2A$ as $A + A$, $3A$ as $A + A + A$ and so on.

So $nA = A + A + A + \ldots n$ times

The same rule applies to matrices also.

If $A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$

$$3A = A + A + A$$

$$= \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+2 & 3+3+3 \\ 5+5+5 & 7+7+7 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 9 \\ 15 & 21 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 2 & 3 \times 3 \\ 3 \times 5 & 3 \times 7 \end{pmatrix}$$

We can now make a general definition of **scalar multiplication**. If A is a matrix, then $3A$ is also a matrix in which each entry of A gets multiplied by 3.

Example :

If $X = \begin{pmatrix} 4 & 2 \\ -5 & 6 \end{pmatrix}$, find $5X$.

Solution :

$$X = \begin{pmatrix} 4 & 2 \\ -5 & 6 \end{pmatrix}$$

$$5X = 5\begin{pmatrix} 4 & 2 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} 5 \times 4 & 5 \times 2 \\ 5 \times -5 & 5 \times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 10 \\ -25 & 30 \end{pmatrix}$$

Exercise 6-5

1. If $A = \begin{pmatrix} 4 & -1 \\ 5 & 7 \end{pmatrix}$, find $5A$.

2. If $X = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$, find $2X$, $3X$ and $7X$.

Simplify :

3. $2(5 \quad 4 \quad 3) + 3(-4 \quad 2 \quad 1)$

4. $3\begin{pmatrix} 1 & -6 \\ 2 & 1 \end{pmatrix} + 7\begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} - 4\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$

5. If $A = \begin{pmatrix} 1 & 3 \\ -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$,

   find the value of $2A + 3B$.

6. Find $A-2B+3C$, if $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 3 & -5 \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} -4 & 5 & 1 \\ 6 & 7 & -2 \end{pmatrix}$$

7. Given $A = \begin{pmatrix} 1 & 5 \\ 6 & -4 \end{pmatrix}$ and $B \begin{pmatrix} 2 & 7 \\ 9 & 1 \end{pmatrix}$,

solve for X if $X + 3A = B$.

ANSWERS

(1) $\begin{pmatrix} 20 & -5 \\ 25 & 35 \end{pmatrix}$. (3) $(-2 \quad 14 \quad 9)$

(5) $\begin{pmatrix} 17 & 3 \\ 10 & 31 \end{pmatrix}$. (7) $\begin{pmatrix} -1 & -8 \\ -9 & 13 \end{pmatrix}$

6-6. Matrices : Multiplication

Multiplying a row matrix by a column matrix

Mr. A purchased 4 apples, 5 oranges and 10 bananas. If an apple costs 70 paise, an orange 30 paise and a banana 10 paise, the total cost of purchase

$$= (4 \times 70) + (5 \times 30) + (10 \times 10) = 530 \text{ paise.}$$

This multiplication may be shown by transposing the cost matrix into a column matrix.

$$(4 \quad 5 \quad 10) \begin{pmatrix} 70 \\ 30 \\ 10 \end{pmatrix} = (4 \times 70 + 5 \times 30 + 10 \times 10)$$

$$= 530$$

This is **matrix multiplication.**

The order of the first matrix is 1×3 and that of the second matrix is 3×1. The order of the product matrix is 1×1.

Exercise 6.6

Find the product matrix (if defined)

1. $\begin{pmatrix} 4 & 7 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

2. $\begin{pmatrix} 5 & -2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$

3, $\begin{pmatrix} 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}$

4. $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

5. $\begin{pmatrix} 4 & 5 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ x \end{pmatrix} = 19$ find the value of x.

6, If $\begin{pmatrix} 5 & x & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = 35$, find x.

ANSWERS

(1) (33) (4) $(ax + by + cz)$ (5) $x = 2$ (6) $x = 4$

6-7. Multiplication of two Matrices

The following matrix gives three articles purchased by two persons.

$$\begin{pmatrix} 2 & 5 & 8 \\ 4 & 2 & 3 \end{pmatrix}$$

If the the articles cost Rs. 10, Rs. 15 and Rs. 20 respectivety, the total cost of purchase by the first person is

$$(2 \times 10) + (5 \times 15) + (8 \times 20) = \text{Rs. } 255$$

The total cost of purchase by the second persons is

$$(4 \times 10) + (2 \times 15) + (3 \times 20) = \text{Rs. } 130$$

These may be shown as follows :

$$\begin{pmatrix} 2 & 5 & 8 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ 15 \\ 20 \end{pmatrix} = \begin{pmatrix} 2 \times 10 + 5 \times 15 + 8 \times 20 \\ 4 \times 10 + 2 \times 15 + 3 \times 20 \end{pmatrix}$$

$$= \begin{pmatrix} 255 \\ 130 \end{pmatrix}$$

If the same articles cost Rs. 12, Rs 16 and Rs. 18 in another shop, the total cost may be written as

$$\begin{pmatrix} 2 & 5 & 8 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 12 \\ 16 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \times 12 + 5 \times 16 + 8 \times 18 \\ 4 \times 12 + 2 \times 16 + 3 \times 18 \end{pmatrix}$$

$$= \begin{pmatrix} 248 \\ 134 \end{pmatrix}$$

The above two particulars can be put together and written as

$$\begin{pmatrix} 2 & 5 & 8 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 10 & 12 \\ 15 & 16 \\ 20 & 18 \end{pmatrix} = \begin{pmatrix} 255 & 248 \\ 130 & 134 \end{pmatrix}$$

The second matrix on the left side is cost matrix.

If the cost of a particular article is not given, can we find the total cost of purchase? No, we cannot.

So, we can perform the multiplication of two matrices only when, the number of columns of first matrix is equal to the number of rows of the second matrix.

In the above example, the order of the first matrix is 2×3 and that of the second is 3×2 and the order of their product is 2×2.

Hence, we observe the following :

In matrix multiplication :

(1) The number of columns of the first matrix must be equal to the number of rows of the second matrix.

(2) Multiplication of a matrix of order $l \times m$ with that of a matrix of order $m \times p$ gives a matrix of order $l \times p$.

(3) Each entry in the rows of the first matrix is multiplied with corresponding entry of each column of the second matrix; and the resulting terms are added to form the terms of the product of the two matrics.

For example

$$\begin{pmatrix} 2 & 5 & 8 \\ 4 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 10 & 12 \\ 15 & 16 \\ 20 & 18 \end{pmatrix}$$

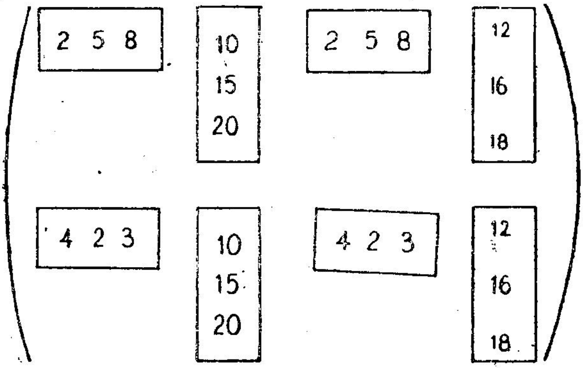


Fig. 6-1

$$= \begin{pmatrix} 2\times 10 + 5\times 15 + 8\times 20 & 2\times 12 + 5\times 16 + 8\times 18 \\ 4\times 10 + 2\times 15 + 3\times 20 & 4\times 12 + 2\times 16 + 3\times 18 \end{pmatrix}$$

$$= \begin{pmatrix} 255 & 248 \\ 130 & 134 \end{pmatrix}$$

Example :

$$\begin{pmatrix} 4 & 5 & -3 \\ 2 & -4 & 6 \\ -1 & 6 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & -2 \\ 0 & 0 \end{pmatrix}$$

The number of columns of the first matrix is equal to the number of rows of the second matrix, each of which is 3. So we can multiply them The product is

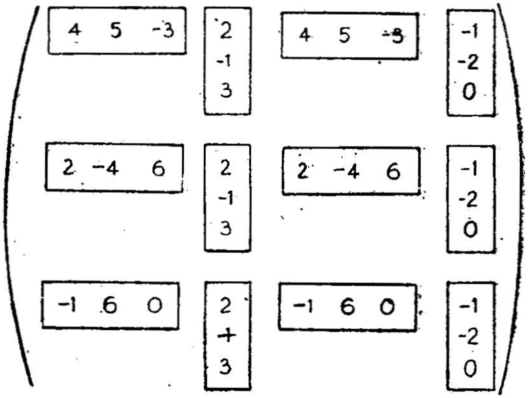


Fig. 6-2.

$$= \begin{pmatrix} 4\times2 + 5\times(-1) + (-3)\times3 & 4\times(-1) + 5\times(-2) + (-3)\times0 \\ 2\times2 + (-4)\times(-1) + 6\times3 & 2\times(-1) + (-4)\times(-2) + 6\times0 \\ (-1)\times2 + 6\times(-1) + 0\times3 & (-1)\times(-1) + 6\times(-2) + 6\times0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -14 \\ 26 & 6 \\ -8 & -11 \end{pmatrix}$$

Exercise 6-7

1. Can we multiply the following matrices ?

	Order of of the first matrix	Order of the second matrix
(a)	2×3	3×1
(b)	1×2	2×2
(c)	3×2	1×2
(d)	2×3	2×3
(e)	3×2	2×3
(f)	2×2	2×3
(g)	1×3	3×2

2. If the product of the matrices A and B is matrix C, fill in the blanks.

No.	Order of A	Order of B	Order of C
(a)	2×3	3×5	—
(b)	3×4	4×2	—
(c)	4×3	—	4×2
(d)	5×4	—	5×1
(e)	—	3×5	1×5
(f)	—	2×3	5×3

(3) Find the order of the matrix B given that

$$(3 \quad 4 \quad 7)\ B = (5 \quad 6).$$

(4) Given that $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 7 \\ 7 & 11 \\ 5 & 6 \end{pmatrix}$

find the order of AB.

Find the product matrix :

(5) $\begin{pmatrix} -3 & 5 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & -2 \end{pmatrix}$

(6) $\begin{pmatrix} 1 & 3 & -4 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

(7) $\begin{pmatrix} 1 & 3 & -3 \\ 5 & 2 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \end{pmatrix}$

(8) $\begin{pmatrix} 1 & 2 \\ 4 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ -5 & 1 & 2 & 2 \end{pmatrix}$

(9) $\begin{pmatrix} 4 & 3 & -2 \\ 5 & 1 & 6 \\ 7 & 2 & 9 \end{pmatrix} \begin{pmatrix} 2 & 5 & 4 \\ 6 & 8 & -1 \\ 5 & -4 & 3 \end{pmatrix}$

(10) Find AB, BA and show that $AB \neq BA$, if

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 4 & -2 & 7 \\ 6 & 1 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -7 & 1 \\ 6 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

(11) If $A = \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 4 \\ 5 & 2 \end{pmatrix}$

find (a) AB (b) BA (c) BC (d) CB (e) AC (f) CA (g) A^2 (h) B^2 (i) C^2

(12) Show that $(A + B)^2 \neq A^2 + 2AB + B^2$

if $A = \begin{pmatrix} 4 & 3 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

ANSWERS

(1) c, d cannot (2) (a) 2×5 (c) 3×2 (e) 1×3

(3) 3×2 (5) $\begin{pmatrix} 16 & -22 \\ 22 & 12 \end{pmatrix}$ (7) $\begin{pmatrix} -8 & -6 \\ 44 & 72 \end{pmatrix}$

(9) $\begin{pmatrix} 16 & 52 & 7 \\ 46 & 9 & 37 \\ 71 & 15 & 53 \end{pmatrix}$ (12) $(A + B)^2 = \begin{pmatrix} 68 & 52 \\ 104 & 81 \end{pmatrix}$

$$A^2 + 2AB + B^2 = \begin{pmatrix} 72 & 43 \\ 123 & 77 \end{pmatrix}$$

6-8. Matrix : Inverse

Unit Matrix

In the system of matrices the unit matrix plays a role similar to that of the number 1 in the real number system.

In R, $x \times 1 = x$. Similarly in matrices, the product of any matrix with the Unit Matrix gives the same matrix. Unit matrices are square matrices and have 1 in their main diagonal and 0 in all other places.

For example,

$$\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$$

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix of order 2×2. The unit matrix is denoted by I.

The unit matrix of order 3×3 is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find the product of the following :

(1) $\begin{pmatrix} 5 & 1 \\ 9 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -9 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 4 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(4) $\begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

$$= -4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We observe from the above the following :

1. The elements along the main diagonal of the Ist matrix are interchanged in the 2nd matrix.

2. The elements along the other diagonal of the Ist matrix are changed in sign in the 2nd matrix.

3. Their product is the unit matrix or a scalar multiple of the unit matrix.

We also observe in example 3,

(The product of the elements along the main diagonal)
– (The product of elements along the leading diagonal)
must be $(4 \times 5) - (7 \times 2) = 6$

In example 4,

$$[3 \times (-2)] - [2 \times (-1)] = -4$$

6, − 4 are called the determinant values of the matrices. We observe that

(1) $$\begin{pmatrix} 4 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & \frac{-2}{6} \\ \frac{-7}{6} & \frac{4}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and (2) $$\begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \frac{-2}{-4} & \frac{-2}{-4} \\ \frac{1}{-4} & \frac{3}{-4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The second matrix is the inverse of the first. Hence to find the inverse of a matrix of order 2×2,

1. Interchange the elements along the main diagonal.
2. Change the sign of the elements along the other diagonal.
3. Find the determinant value of the matrix.
4. Divide each term of the matrix obtained by the determinant value of the matrix.

To find the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Step 1 : $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$

Step 2 : $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Step 3 : $ad - bc$

[Note the determinant values of the original matrix and the transformed matrix are the same.]

Step 4 : $$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

The inverse matrix of A, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is denoted by A^{-1} and is given by

$$A^{-1} = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

We find that $AA^{-1} = A^{-1}A = I$;

When the determinant value of a matrix, $ad - bc = 0$, there is no inverse matrix.

Example :

Find the inverse of $\begin{pmatrix} 4 & 7 \\ -3 & 2 \end{pmatrix}$

Solution :

Step 1 : $\begin{pmatrix} 2 & 7 \\ -3 & 4 \end{pmatrix}$

Step 2 : $\begin{pmatrix} 2 & -7 \\ 3 & 4 \end{pmatrix}$

Step 3 : $(4 \times 2) - (-3) \times (7) = 8 + 21 = 29$

Step 4 : $\begin{pmatrix} \dfrac{2}{29} & \dfrac{-7}{29} \\ \dfrac{3}{29} & \dfrac{4}{29} \end{pmatrix}$

The inverse matrix of $\begin{pmatrix} 4 & 7 \\ -3 & 2 \end{pmatrix}$ is $\begin{pmatrix} \dfrac{2}{29} & \dfrac{-7}{29} \\ \dfrac{3}{29} & \dfrac{4}{29} \end{pmatrix}$

Exercise 6-8

Find the inverses of the following matrices, if they exist.

(1) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (3) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(4) $\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}$ (5) $\begin{pmatrix} -6 & -5 \\ 4 & 2 \end{pmatrix}$ (6) $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

(7) $\begin{pmatrix} 4 & 7 \\ 1 & 3 \end{pmatrix}$ (8) $\begin{pmatrix} -6 & 2 \\ 7 & 3 \end{pmatrix}$ (9) $\begin{pmatrix} 5 & -3 \\ 1 & 4 \end{pmatrix}$

(10) $\begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix}$

ANSWERS

(1) $\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$ (4) $\begin{pmatrix} \frac{-3}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{-3}{16} \end{pmatrix}$

(7) $\begin{pmatrix} \frac{3}{5} & \frac{-7}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{pmatrix}$

(6), (10) inverse matrix does not exist.

6-9 Matrices : Solution of Equations

Let us now use matrices for solving linear equations in two variables.

Example 1 :

Solve :

$$3x + 5y = 29$$
$$x + 2y = 11$$

Solution :

$$3x + 5y = 29$$
$$x + 2y = 11$$

These can be written as a product of two matrices.

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29 \\ 11 \end{pmatrix} \qquad (1)$$

Multiplying equation (1) on both sides by the inverse matrix of

$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ which is $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$, we get

$$\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 29 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 58 - 55 \\ -29 + 33 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$x = 3, y = 4$

The solution set is { (3, 4) }

Example 2 :

Solve using matrices.

$$4x - 3y = 14$$
$$5x + 2y = 29$$

Solution :

Writing the equations

$$4x - 3y = 14$$
$$5x + 2y = 29$$

as a product of two matrices, we get

$$\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 29 \end{pmatrix} \qquad (1)$$

The inverse of $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$ is $\begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{4}{23} \end{pmatrix}$

Pre-multiplying (1) by $\begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{4}{23} \end{pmatrix}$

We get $\begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{4}{23} \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{4}{23} \end{pmatrix} \begin{pmatrix} 14 \\ 29 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{28}{23} + \frac{87}{23} \\ \frac{-70}{23} + \frac{116}{23} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{115}{23} \\ \frac{46}{23} \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$x = 5, \quad y = 2.$

Solution set is $\{ (5, 2) \}$

Exercise 6-9

Solve the following sets of equations using matrices.

(1) $3x + 2y = 5$
$7x + 3y = 10$

(2) $4x + 3y = 17$
$3x + 4y = 18$

(3) $2x + y = 6$
$5x - 3y = 15$

(4) $3x - 5y = 8$
$7x + 2y = 5$

(5) $x - 2y = -8$
$3x + 4y = 26$

(6) $4x - 7y = -53$
$7x - 2y = -21$

(7) $4x - 5y = -6$
$5x - 2y = 1$

(8) $2x - 3y = 1$
$x + 4y = 6$

(9) $3x + 2y = -13$
$x - 4y = 5$

(10) $5x - 7y = 11$
$2x - 3y = 5$

ANSWERS

(1) { (1, 1) }	(2) { (2, 3) }	(3) { (3, 0) }
(4) { (1, −1) }	(5) { (2, 5) }	(6) { (−1, 7) }
(7) { (1, 2) }	(8) {(2, 1) }	
(9) { (−3, −2) }	(10) { (−2, −3) }	

REVIEW

1. A rectangular arrangement of information is called a matrix and it helps to present in a concise form numerical data.

2. A matrix in which all the entries are zero, is called a zero matrix.

3. Two matrices of the same order are said to be equal when the corresponding elements in the matrices are equal.

4. We define addition only for matrices of the same order. In addition, the corresponding entries are added.

5. A matrix formed by the additive inverses of the elements of a matrix is called the negative of the original matrix.

6. $A - B$ is defined as $A + (-B)$.

7. In scalar multiplication, all the entries of a matrix are multiplied by a real number.

8. We define multiplication only where the number of columns of the first matrix equals the number of rows of the second matrix. We multiply each entry in the row of the first matrix with the corresponding entry of each column of the second matrix and add the resulting terms.

9. The determinant value of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $(ad - bc)$.

10. The inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{pmatrix} \frac{d}{ab-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ab-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

SELF ASSESSMENT TEST

1. Simplify :

$$\begin{pmatrix} 1 & 2 & -3 \\ 5 & 1 & 6 \\ 7 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 5 \\ 6 & 2 & 0 \\ 5 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ -4 & 5 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

2. Given that

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix} B = \begin{pmatrix} 5 & -2 \\ 6 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$$

find $A - 2B + 3C$.

3. Solve :

$$\begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{pmatrix} + X = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & 1 & 4 \end{pmatrix}$$

4. Find the product of

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & 3 \\ 5 & 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 2 & 1 \\ 7 & 6 \end{pmatrix}$$

5. Find the inverse of

$$\begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$$

6. Solve using matrices :

$$2x + 3y = 13$$
$$3x + 2y = 12$$

EULER (1707 — 1783)

In the last lesson, we learnt about $F + V = E + 2$. This was discovered by the great mathematician Leonard Euler, a Swiss by birth, who lived about two hundred years ago. For the past fifty years, the publication of his works is going on and it is yet to be completed. From this, one can very easily understand his vast contribution to Mathematics.

Euler was born in April 1707. His father was a pastor and he wanted his son too to become a priest. But, the love he bore for Mathematics changed his life.

Euler had his education at Brussels University, his favourite subjects being Mathematics and Hebrew. While a student there, he came in contact with Johannes Bernoulli, whose family had the credit of giving this world eight great Mathematicians in three successive generations. Bernoulli, the Senior came forward to teach Euler Mathematics for a week Apart from the senior Bernoulli, Euler cultivated friendship with David, Nicholas and others belonging to that renowned family and hence came to be considered as one of its members.

In those days, the Academy in St. Petersburg in Russia was well-known for its research. Euler joined the faculty to study medicine. He studied about ears and during the course of his researches, he formulated many laws of acoustics and expounded many theories regarding sound. By his contribution to medical research, he proved to the world that he was capable of doing research in any field of knowlodge.

When Russia came under the tyranny of the Tzar, Euler accepted the offer to join the Berlin University. The Prussian Queen wanted to get his friendship. But Euler gave mono-syllable responses to her queries. "Why do you evade talking with me, Mr. Euler?", he was asked. He answered, "Your Majesty, I have come here from the land where one is hanged if one speaks ", This reply clearly shows us how sick he felt over the prevailing situation in his country then.

The Prussian King Fredrick utilized Euler's services for the welfare of his kingdom. With his help, various reforms in monetary system, irrigation, river-transport, pension scheme, etc., were introduced and they bear a witness to his wonderful intellect The Government Employees' Pension Scheme, now in force, in most of the countries of the world is formulated on the lines of Euler's scheme.

Since Euler was lacking in the art of flattery in courts, he soon fell into disfavour with Fredrick. When Euler sensed the change of atmosphere in the court, he went again to St. Petersburg, the present Leningrad and continued his research.

It was the good reign of the Empress Tzarina Catherine and she graciously looked after the large eighteen – member family of Euler. Euler had already lost one eye and the second eye also became blind. But his complete blindness, in no way impaired his intellect in the least. Even as a boy, he had the habit of memorizing many great epics and this power of memory was of immense help to him now Once two of his disciples were working at a problem and differed in the fiftieth digit in the solution When this was brought to Euler's notice, he worked it out mentally and gave out the correct answer. "A person's physical deformity will in no way impair one's achievement, if there is a will." This maxim was well brought out in Euler's life. Further, from this incident his power of retention and indomitable will power are also known to the world.

The world of mathematics is enriched by his contributions. It was he who found out the relations among the vertices, edges and faces of solids; he also solved the celebrated 'Konigsberg puzzle'. Just as his life had great significance, so also was his death significant.

One evening, after solving some new problems pertaining to space travel in balloons, he had his supper with his friends, played for a while with his grandchildren, then placing his pipe down said 'I am dying' and he breathed his last.

Leonard Euler had no sight but gave intellectual light. His life was a mixture of joy and sorrow. In spite of all this, he threw his heart and soul in his research. He derived inspiration from his belief that he was born with a purpose.

All the incidents in his life teach us valuable lessons. Let his life teach us to strive to cultivate the memory power and, the will power to perform great things and to achieve our goals in our walk of life. If this is achieved, no doubt, the future will be a golden age for humanity.

7. GEOMETRY

7-1. Revision

7-1-1. Congruent traiangles

A triangle is the union of three segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ drawn by joining three non-collinear points A, B, C in a plane. This is denoted by $\triangle ABC$. Let us now recall the properties of triangles we have already learnt.

Figures which are equal in all respects are said to be congruent.

(i) In $\triangle ABC$ and $\triangle DEF$, if

$\overline{AB} \equiv \overline{DE}$

$\overline{AC} \equiv \overline{DF}$

$\angle A \equiv \angle D$

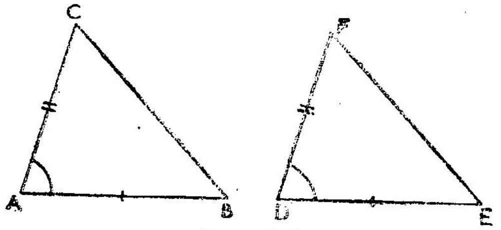


Fig. 7-1.

then $\triangle ABC \equiv \triangle DEF$.

This is denoted by 'Side-Angle-Side' or simply SAS.

Note :

This is valid only if the angle included between the equal sides are equal. If $\overline{AB} \equiv \overline{DE}$, $\overline{AC} \equiv \overline{DF}$, $\angle C \equiv \angle F$, then the triangles need not be congruent.

(ii) In $\triangle LMN$ and $\triangle PQR$, if

$\angle L \equiv \angle P$

$\angle M \equiv \angle Q$

$\overline{LM} \equiv \overline{PQ}$

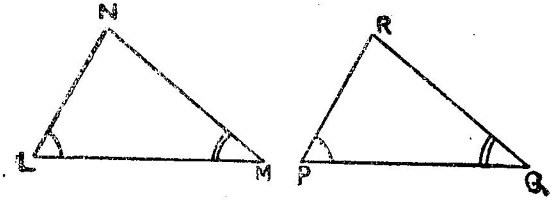


Fig. 7-2.

then $\triangle LMN \equiv \triangle PQR$.

This is denoted by 'Angle-Side-Angle' or simply ASA.

(iii) In $\triangle XYZ$ and $\triangle X'Y'Z'$, if

$\overline{XY} \equiv \overline{X'Y'}$

$\overline{YZ} \equiv \overline{Y'Z'}$

$\overline{ZX} \equiv \overline{Z'X'}$

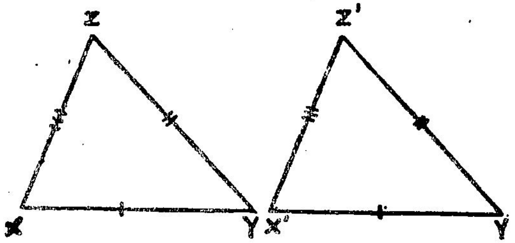


Fig. 7-3.

Then $\triangle XYZ \equiv \triangle X'Y'Z'$

This is denoted by 'Side-Side-Side' or simply SSS

(iv) In $\triangle ABC$ and $\triangle PQR$, if

$$m\angle A = m\angle P = 90°$$
$$\overline{AC} \cong \overline{PR}$$
$$\overline{BC} \cong \overline{QR}$$

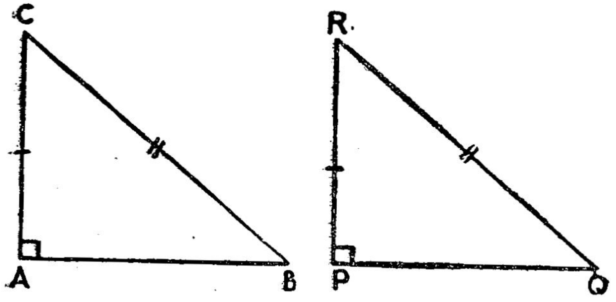


Fig. 7-4.

then right $\triangle ABC \cong$ Right $\triangle PQR$.

This is denoted by **'Hypotenuse-Leg'** or simply **HL.**

Exercise 7-1-1

Which of the following pairs of traingles are congruent? (Equal sides and equal angles are marked in the figure.)

(1)

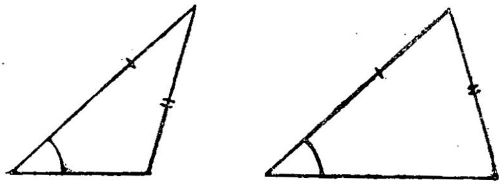


Fig. 7-5.

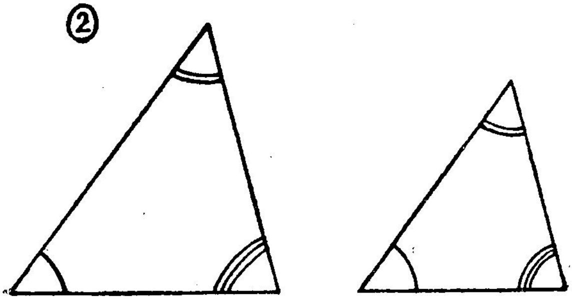


Fig. 7-6.

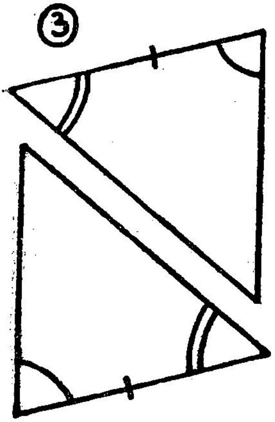


Fig. 7-7.

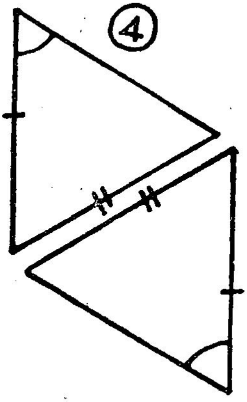


Fig. 7-8.

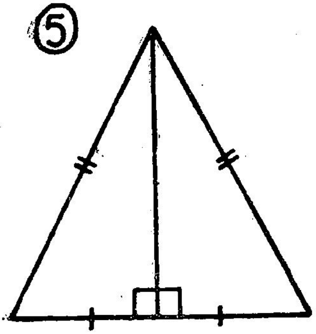


Fig. 7-9.

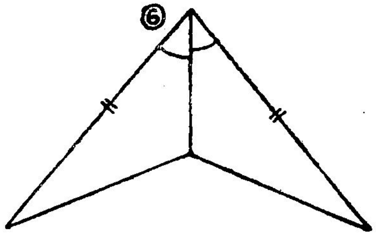


Fig. 7-10.

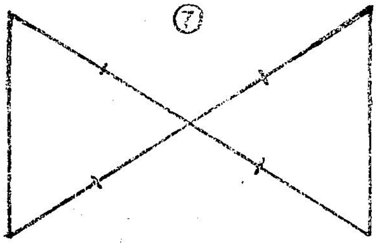


Fig. 7-11.

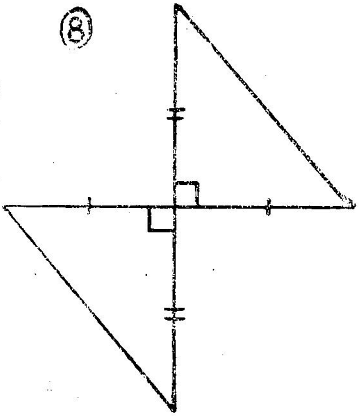


Fig. 7-12.

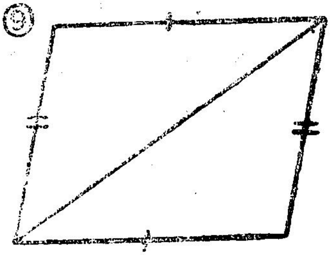


Fig. 7-13.

ANSWERS

(1) No (2) No (3) Yes, ASA (4) No (5) Yes, HL (6) Yes, SAS (7) Yes, SAS (8) Yes, SAS (9) Yes, SSS.

7-1-2. Concurrent Lines

If three or more lines meet at a common point, they are said to be concurrent. Now let us study about a few concurrent lines in a triangle.

1. In $\triangle ABC$, perpendiculars $\overline{AL}$, $\overline{BM}$, $\overline{CN}$ are drawn from the vertices A, B and C. They meet at O, the **Orthocentre** of $\triangle ABC$

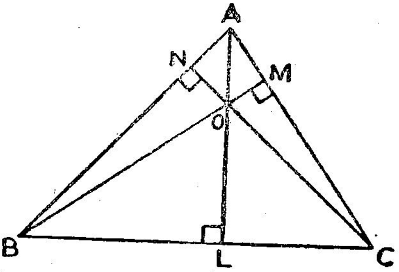


Fig. 7-14.

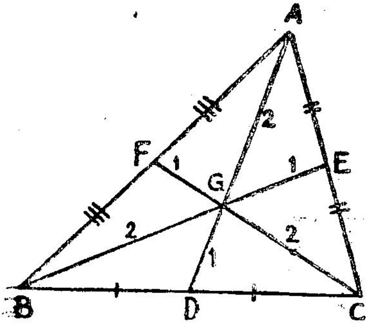


Fig. 7-15.

2. In $\triangle ABC$,
if $BD = DC$
$CE = EA$
$AF = FB$

$\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are **medians.**

These three medians meet at G.

Further, $AG : GD = BG : GE = CG : GF = 2 : 1$

G is said to be the **centroid** of the triangle. (You may have studied in your science class that G is the C. G. of the triangular lamina ABC).

3. $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$ are respectively the **bisectors** of $\angle A$, $\angle B$ and $\angle C$. These three segments pass through the point I. Draw $\overline{ID} \perp \overline{BC}$. The circle drawn with I as centre and ID as radius will touch the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. Hence, I is said to be the **Incentre** of the triangle.

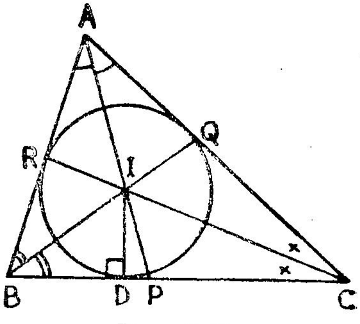


Fig. 7-16.

4. The **perpendiculur bisectors** of the three sides of a triangle pass through a point. This point, S is called the **circumcentre** of the triangle.

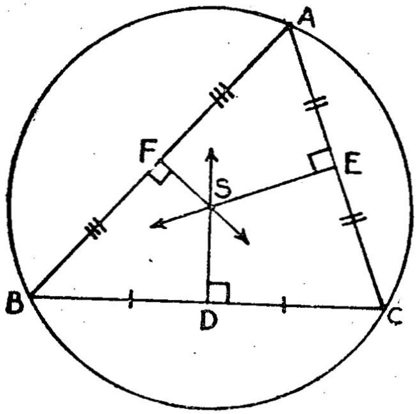


Fig 7-17.

In $\triangle ABC$, D, E, F are the midpoints of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

Further, $\overline{DS} \perp \overline{BC}$

$\overline{ES} \perp \overline{CA}$

$\overline{FS} \perp \overline{AB}$

The segments $\overline{DS}$, $\overline{ES}$, $\overline{FS}$ meet at S. With S as centre and SA as radius, draw a circle. This circle passes through A, B, C.

Exercise 7-1-2

Find the following through **paper folding**.

1. The perpendicular drawn from the midpoint of the base of an isosceles triangle bisects the vertical angle.
2. The perpendicular drawn from the vertex of an isosceles triangle bisects the base.
3. The Circumcentre, the Orthocentre and the Incentre of an isosceles triangle are Collinear.
4. The Circumcentre, the Orthocentre and the Incentre of an equilateral triangle are one and the same point.
5. The Circumcentre and the Orthocentre of an Obtuse-triangle lie outside the triangle.

7-1-3. Circles

The word 'circle' has been defined to mean a certain simple closed curve in a plane. The term is however, often used to indicate the part of the plane inside the

boundary of this curved line. Let us recall the facts that we have already learnt about chords, arcs and angles of a circle.

(1) Equal chords are equidistant from the centre.

O is the centre of the circle.

$\overline{AB} \equiv \overline{CD} \equiv \overline{EF}$

If $\overline{OP} \perp \overline{AB}$, $\overline{OQ} \perp \overline{CD}$,

and $\overline{OR} \perp \overline{EF}$, then

$OP = OQ = OR$

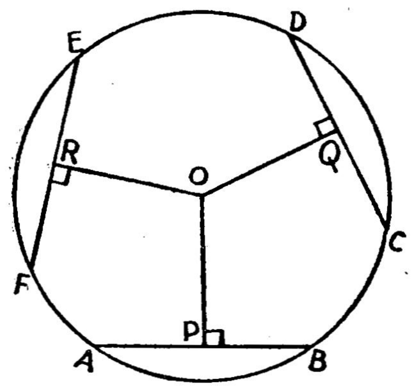


Fig. 7-18.

Conversely, chords that are equidistant from the centre are equal.

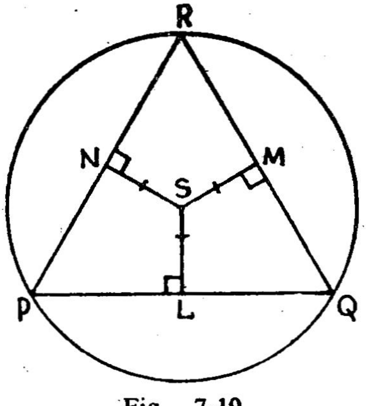


Fig 7-19.

S — Centre of the circle.

$\overline{SL} \perp \overline{PQ}$

$\overline{SM} \perp \overline{QR}$

$\overline{SN} \perp \overline{RP}$

If $SL = SM = SN$, then

$\overline{PQ} \equiv \overline{QR} \equiv \overline{RP}$

(2) If chords are equal they cut off equal arcs.

In a circle with O as Centre,

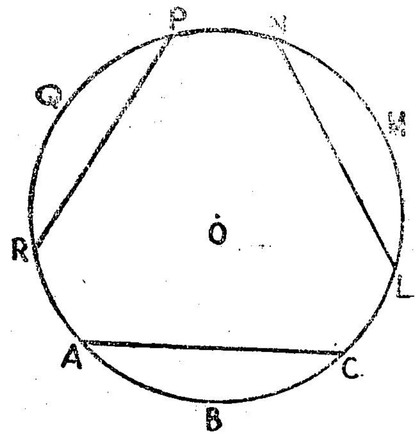


Fig. 7-20.

If $\overline{AC} \equiv \overline{LN} \equiv \overline{PR}$, then

$\overset{\frown}{ABC} \equiv \overset{\frown}{LMN} \equiv \overset{\frown}{PQR}$

Conversely, if arcs are equal, the chords of the arcs are equal.

If $\overset{\frown}{APB} \equiv \overset{\frown}{BQC} \equiv \overset{\frown}{CRA}$,

then

$\overline{AB} \equiv \overline{BC} \equiv \overline{CA}$

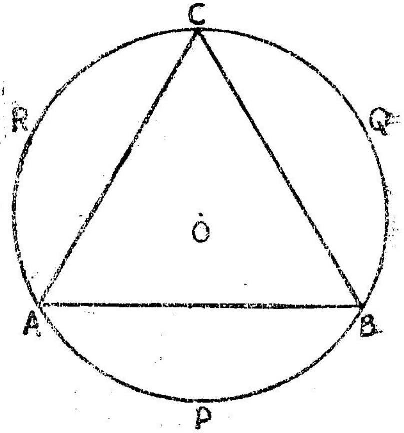


Fig. 7-21.

(3) The angle in a semi-circle is a right angle.

If $\overline{AOB}$ is a diameter,

$m\angle ACB = m\angle ADB$

$= m\angle AEB = 90°$.

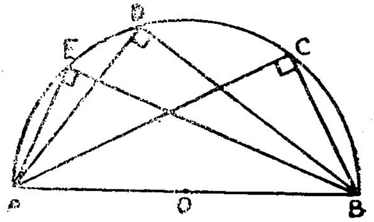


Fig. 7-22.

(4) The opposite angles of any quadrilateral inscribed in a circle are supplementary.

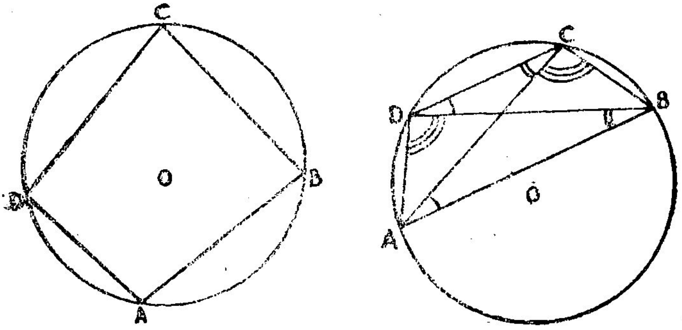


Fig. 7-23.

If *ABCD* is a cyclic quadrilateral

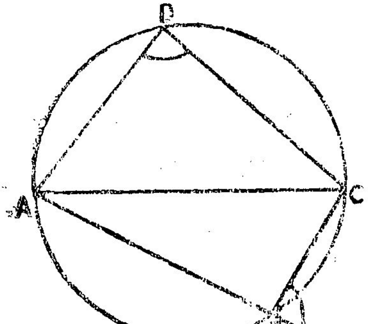


Fig. 7-24.

$m\angle A + m\angle C$

$= m\angle B + m\angle D = 180°$

Also in quadrilateral *ABCD*, if $\overline{AB}$ is extended to *E*,

$m\angle D = m\angle CBE$

$= m\angle BAC + m\angle BCA$

Exercise 7-1-3

1. In a circle of diameter 20 cm a chord and a diameter have common end - points. The length of the chord is 16 cm. Find the distance of the chord from the centre.

2. A tangent is drawn to a circle. A chord drawn parallel to the tangent bisects the radius drawn through the point of contact. The length of the chord is 18 cm. Find the radius of the circle.

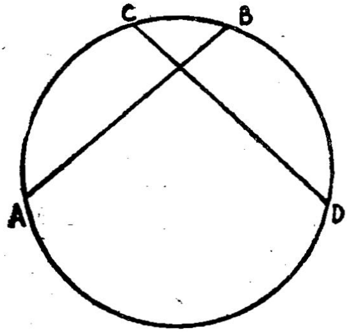


Fig. 7-25.

3. In fig. 7-25, if $\overline{AB} \cong \overline{CD}$, what arc is congruent to $\widehat{AC}$?

4. $\overline{AB}$ is a diameter of a circle. Points C and D are on the circle on the opposite sides of AB such that $BC = BD$ Are the triangles $\triangle ABC$ and $\triangle ABD$ congruent ? Why ?

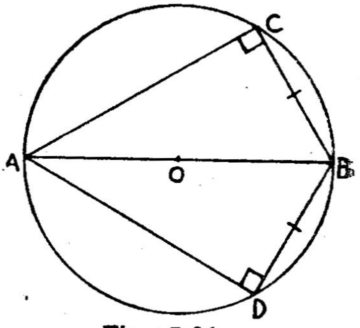


Fig. 7-26.

5. In a semi - circle $\widehat{ACB}$, from the centre O, perpendiculars $\overline{OD}$ and $\overline{OE}$ are drawn to $\overline{AC}$ and $\overline{CB}$. Find the measure of the $\angle DOE$.

6. In a cyclic quadrilateral $ABCD$, find the angle of the quadrilateral which is equal to $m\angle DCA + m\angle DAC$.

ACTIVITY

Draw a cirlce. Draw chords of equal lengths and mark their midpoints. These points lie in a circle with the same centre as the first circle.

ANSWERS

(1) 6 cm (2) $6\sqrt{3}$ (3) $\widehat{BD}$ (4) Yes, HL
(5) 90° (6) $m\angle B$

7-2. Some Geometrical Concepts

Let us learn some more geometrical concepts this year through paper folding or paper cutting. Let us verify our results by drawing figures and by calculation.

7-2-1. Similar Triangles

The simplest plane figure is triangle.

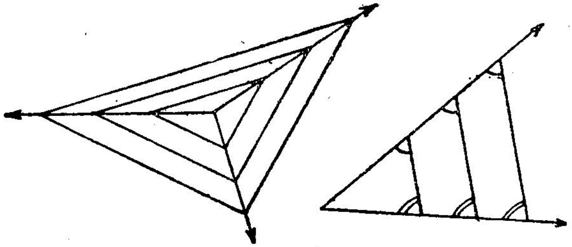


Fig. 7-27.

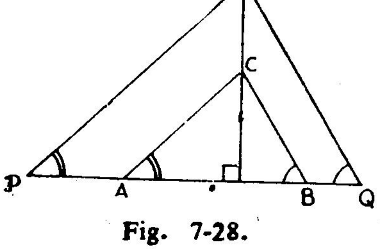


Fig. 7-28.

Draw a triangle ABC. Measure its sides. Draw another triangle PQR such that its sides measure twice that of $\triangle ABC$. Measure the corresponding angles of these two triangles. What do you learn ?

Let $\triangle$ ABC, $\triangle$ PQR be similar.

We write this as $\triangle ABC \sim \triangle PQR$.

$m\angle A = m\angle P$; $m\angle B = m\angle Q$; $m\angle C = m\angle R$

$AB : PQ = BC : QR = CA : RP$

i.e., $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CA}{RP}$

In the same way, draw triangle whose sides are (a) three times the given triangle, (b) half the measure of the given triangle. Measure the angles which correspond to each other. Find the ratio of the corresponding sides.

> In two similar triangles, corresponding sides are proportional.

Conversely,

> If two triangles have their corresponding sides proportional, they are similar.

We can arrive at the following from the above concepts.

If in $\triangle ABC$ and $\triangle ADE$

$\angle A \equiv \angle A$

$\angle B \equiv \angle D$

$\angle C \equiv \angle E$,

then $\triangle ABC \sim \triangle ADE$

$\Longrightarrow \dfrac{AB}{AD} = \dfrac{AC}{AE} = \dfrac{BC}{DE}$;

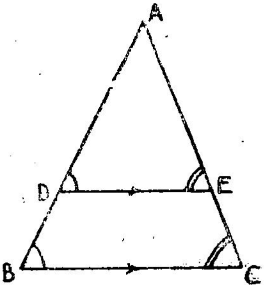


Fig. 7-29.

$$\frac{AB}{AD} - 1 = \frac{AC}{AE} - 1 \text{ (Subtract 1 from each side)}$$

$$\frac{AB - AD}{AD} = \frac{AC - AE}{AE}$$

$$\frac{BD}{DA} = \frac{CE}{EA}$$

In fig. 7-28, let the height of $\triangle ABC$ be h_1 and that of $\triangle PQR$ be h_2.

$$\frac{h_1}{h_2} = \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = k$$

$$\therefore \quad AB = k\,PQ$$

$$\text{also,} \quad h_1 = kh_2$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times AB \times h_1$$

$$= \frac{1}{2} \cdot k \cdot PQ \cdot kh_2$$

$$= \frac{1}{2} k^2 \cdot PQ \cdot h_2$$

$$\text{Area of } \triangle PQR = \frac{1}{2} \times PQ \times h_2$$

$$\therefore \quad \triangle ABC : \triangle PQR = \frac{1}{2} k^2 \cdot PQ \cdot h_2 : \frac{1}{2} PQ \cdot h_2$$

$$= k^2 : 1$$

Hence, if the sides of similar triangles are in the ratio $k : 1$, then their areas are in the ratio $k^2 : 1$.

In general,

> **In similar triangles, if sides are in the ratio $l : m$, then the ratio of their areas is $l^2 : m^2$.**

Exercise 7-2·1

1. In ABC,

   $AB = 9$; $BC = 12$; $CA = 15$

   In $\triangle\ DEF$, $DE = 6$

   If $\triangle\ ABC$ ||| DEF, find EF and FD.

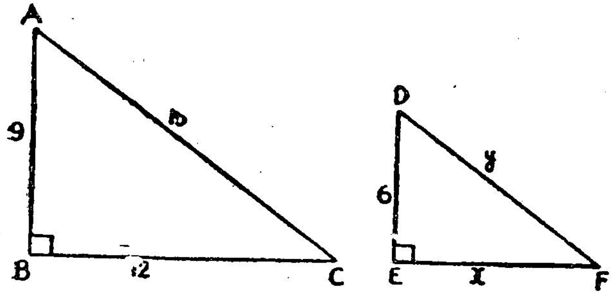


Fig. 7-30.

2. In Fig. 7-31,

   find BC and CE.

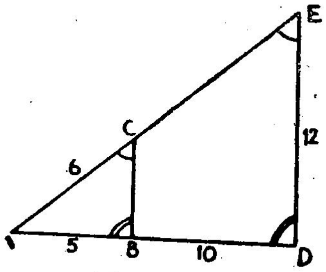


Fig. 7-31.

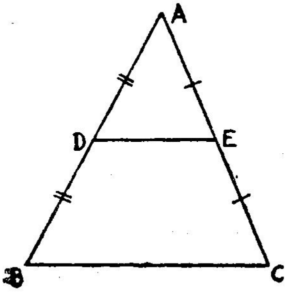


Fig. 7-32.

3. In Fig. 7-32,

   what is DE ?

   Are $\triangle ABC$ and

   $\triangle ADE$ similar ?

4. From Fig. 7-33,

find any two similar Δs.

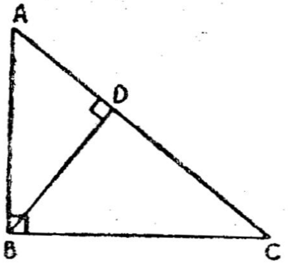


Fig. 7-33.

5. The bases of tw riangles are in the ratio 4 : 7. Find the ratio of their areas.

6. The areas of two similar triangles are in the ratio 4 : 9. Find the ratio of their bases.

ANSWERS

(1) $x = 8$; $y = 10$ (2) 4, 12 (3) $DE = \frac{1}{2} BC$; Yes,

(4) Any two of the triangles $\Delta\ ABC$, $\Delta\ BDC$, $\Delta\ ADB$

(5) 16 : 49 (6) 2 : 3

7-2-2. Similar Figures

A stone thrown in still water produces ripples. The circles thus caused steadily expand into bigger circles in all directions.

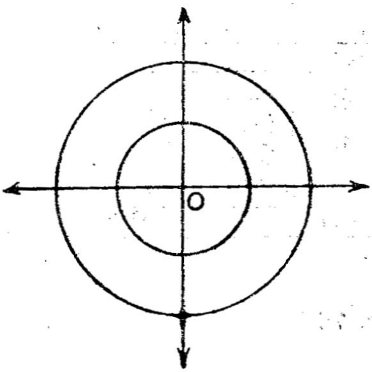


Fig. 7-34.

From the ratio of the radii of circles, we can find the ratio of their areas. For example, if r_1, r_2 are the radii of two circles, then their areas are πr_1^2, πr_2^2 sq. units. Hence if the ratio of the radii is $r_1 : r_2$ the ratio of their areas will be r_1^2, r^2_2.

We know that photographs can be enlarged. A photograph when enlarged, is made bigger in all respects Hence the length of the photograph will vary; but the angles remain constant.

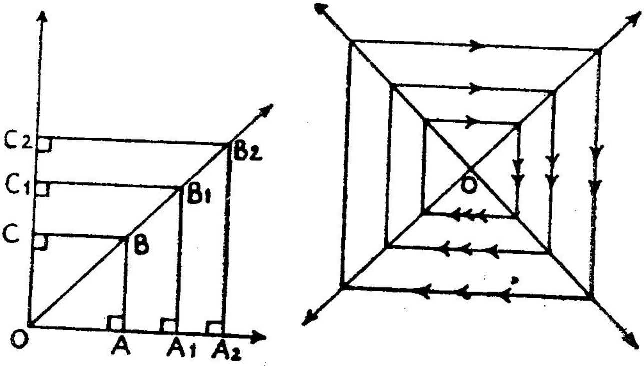


Fig. 7-35.

Hence if two square pictures have sides in the ratio $a_1 : a_2$ their areas will be in the ratio $a_1^2 : a_2^2$. The measures of the angles are always 90°.

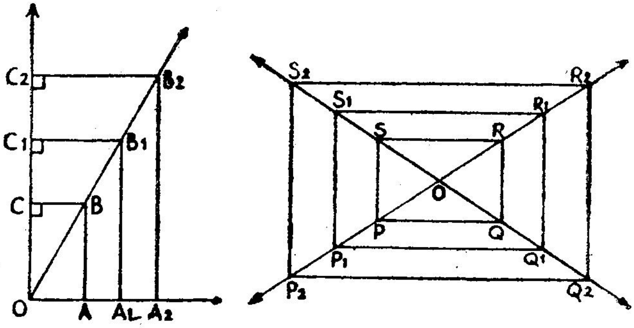


Fig. 7·36

Similar figures such as squares, rectangles and quadrilaterals can be formed as shown in the figures.

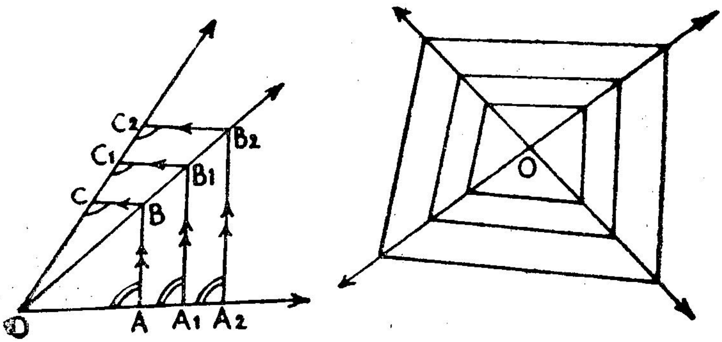


Fig. 7-37

Exercise 7-2-2

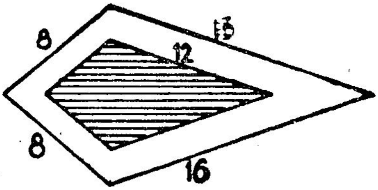


Fig. 7-38

In the adjoining figure the sides of the bigger quadrilateral are given. The smaller quadrilateral is similar to the bigger one. Find the sides of the inner quadrilteral.

2. The length and breadth of a photograph are 4 cm and 3 cm respectively. It is enlarged so that its breadth is 7·5 cm. Find the length of the enlarged figure.

3. The ratio of the sides of two squares is 3 : 5. Find the ratio of their areas.

4. The ratio of the areas of two squares is 16 : 25. Find the ratio of their sides.

5. Find the side of the square whose area is 4 times that of a square of side 4 cm.

6. The length and breadth of a rectangle are 6 cm and 5 cm. Find the area of the rectangle whose sides are twice the sides of the given rectangle.

ANSWERS

(1) 6, 6, 12 (2) 10 cm (3) 9 : 25 (4) 4 : 5

(5) 8 (6) 120 cm^2

7-2-3. Pythagoras Theorem

Draw on a graph sheet a triangle of sides 3, 4, 5 units. The angle contained by the sides of 3 and 4 units is found to be a right angle. Draw a square on each side. Their areas are respectively 9, 16, 25 Sq. units.

$$9 + 16 = 25$$

$$\text{i.e.}\quad 3^2 + 4^2 = 25$$

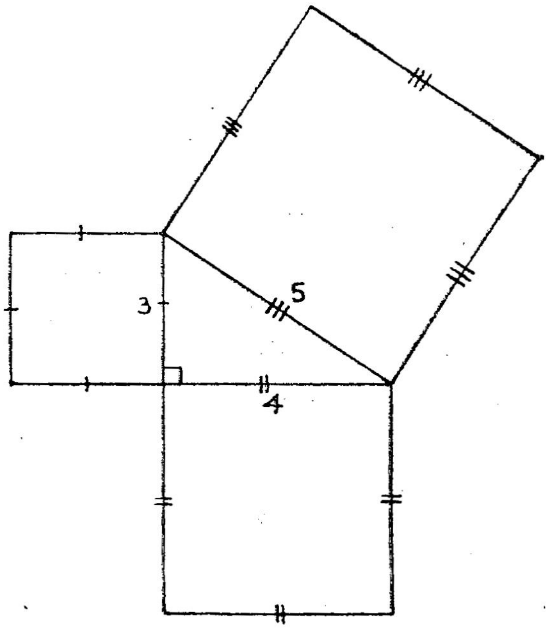


Fig. 7-39.

Repeat the process with sides 5, 12, 13 units. The angle contained by the smaller sides is found to be a right angle.

Also $25 + 144 = 169$

$5^2 + 12^2 = 13^2$

In general, in a $\triangle$ ABC,

if $m\angle A = 90°$,

$BC^2 = AC^2 + AB^2$

$a^2 = b^2 + c^2$

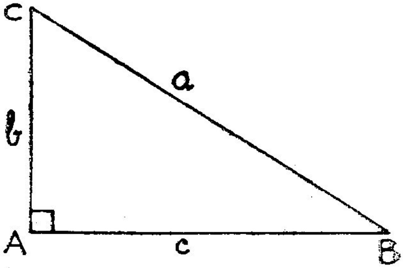


Fig. 7-40.

Hence,

> In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides

This famous theorem was discovered by the Greek mathematician Pythagoras (*c* 582 *BC*). Hence, it is known as the **Pythagoras Theorem.**

The ratios of the sides of certain types of right triangles were known to Babylonians 4000 years ago. The Egyptians built their pyramids with a triangle made of rope with sides in the ratio 3 : 4 : 5 forming a right triangle.

We have learnt the relation between the squares of the sides of a right triangle. Conversely, we can find out whether a given triangle is a right triangle, from the relation between the squares of the sides of the triangle.

For Example

1. Let the measures of the sides of a triangle be 9, 12, 15 units. $9^2 = 81$; $12^2 = 144$; $15^2 = 225$

As $15^2 = 12^2 + 9^2$, the sides of 9 and 12 units contain the right angle. Side of 15 units denotes the hypotenuse.

Hence, 9, 12, 15 are the lengths of sides of a right triangle.

2. In a triangle with sides 8, 15, 17 units,

$8^2 + 15^2 = 17^2$

$64 + 225 = 289$

Hence, sides of 8, 15 contain the right angle. Side of 17 denotes the hypotenuse.

In general, in a triangle if $a^2 = b^2 + c^2$, then a is the hypoteuse, b and c are the sides containing the right angle.

> If in a triangle the square of the length of one side is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle. The angle opposite to the longest side is a right angle.

Exercise 7-2-3

1. Determine whether the triangles with the following measurements of sides are right ... triangle.

   (i) 12, 20, 16
   (ii) 26, 10, 24
   (iii) 8, 15, 16

2. The sides of a right triangle are 3, 4, 5. Find out if triangles having (i) twice (ii) thrice (iii) four times the sides given above are also right triangles.

3. Find the hypotenuse of a right triangle with the following measurements of the sides containing the right traingle.

   (i) 1·5, 2 (ii) 2·5, 6 (iii) 1, $\frac{1}{3}$

4. Find the third side of a right triangle, given the hypotenuse and one side respectively as follows :

   (i) 1·3, 0·5 (ii) 1·5, 1·2 (iii) 1·7, 0·8

ANSWERS

1. (i) Yes (ii) Yes (iii) No 2 Yes

3. (i) 2·5 (ii) 6·5 (ii.) $1\frac{2}{3}$

4. (i) 1·2 (ii) 0·9 (iii) 1·5

7-2.4 Pythagoras Theorem II

Example

In the figure, show that $c^2 = a^2 + b^2 + 2ax$.

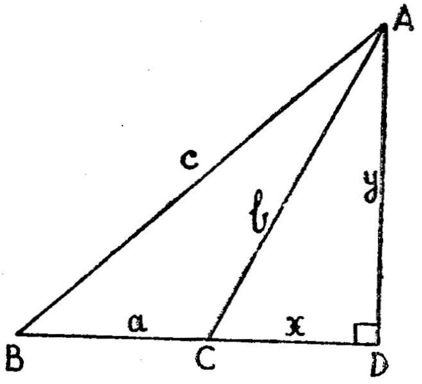


Fig. 7-41.

Solution :

In the right $\triangle$ ADB,

$$AB^2 = AD^2 + DB^2$$

$$AB^2 = AD^2 + (DC + CB)^2$$

$$c^2 = y^2 + (x + a)^2$$

$$c^2 = y^2 + x^2 + 2ax + a^2$$

In the right $\triangle$ ADC, $y^2 + x^2 = b^2$

$\therefore$ $c^2 = a^2 + b^2 + 2ax$

Exercise 7-2·4

1.

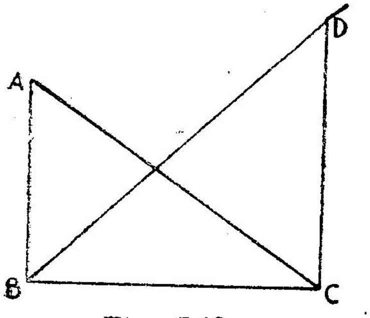


Fig. 7-42.

In fig. 7-42, if $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$, then show that $AC^2 - AB^2 = BD^2 - DC^2$

2.

In fig. 7-43, if $\overline{DB} \perp \overline{AC}$, show that $AD^2 + BC^2 = AB^2 + DC^2$

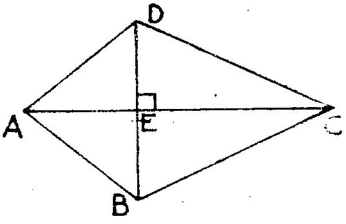


Fig. 7-43.

3.

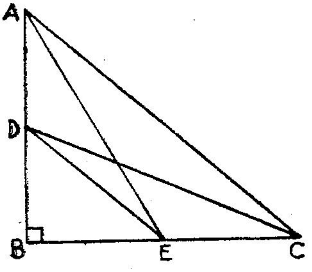


Fig. 7-44.

In fig. 7-44, if $m\angle ABC = 90°$, show that

$AC^2 + DE^2 = AE^2 + DC^2$

4. In fig. 7-45, if $\overline{BD} \perp \overline{AD}$, C is the midpoint of $\overline{AD}$ and $m\angle A = 45°$, show that $AB^2 = 8\ AC^2$

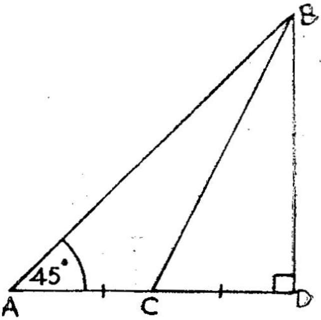


Fig. 7-45

7-2.5 Inscribed angle

Draw a circle on a piece of paper and cut it out. Fold the circular paper along any $\overline{AB}$. Now we get two segments; ACB is known as the **major segment** and ADB the **minor segment**.

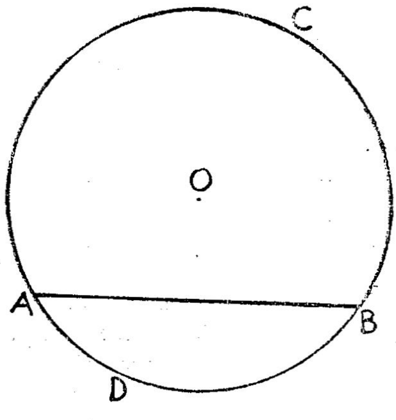


Fig. 7-46.

In the major segment ACB, fold angles $\angle APB$, $\angle AQP$, $\angle ARB$. $\angle ASB$. As shown in Fig 7-47, cut out $\angle P$ and place it on $\angle Q$, $\angle R$, $\angle S$ separately. We find that

$\angle P \equiv \angle Q \equiv \angle R \equiv \angle S$. We can verify this fact by measurement also.

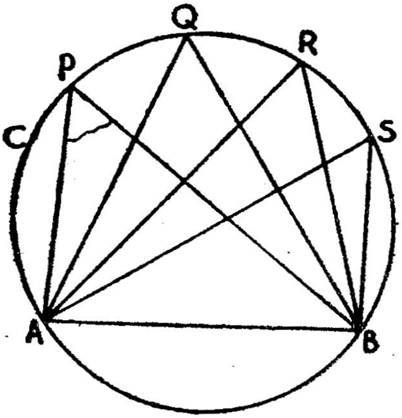


Fig. 7-47.

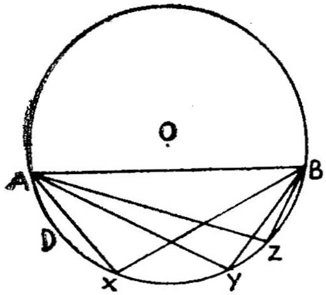


Fig. 7-48.

Draw a circle and mark a chord $\overline{AB}$ in it. In the major segment ACB, measure the angles $\angle APB$, $\angle AQB$, $\angle ARB$, $\angle ASB$ and find whether they are equal.

In the same way, the angles $\angle AXB$, $\angle AYB$, $\angle AZB$ in the minor segment ADB are folded. Cut $\angle X$ and place it on $\angle Y$, $\angle Z$. We find that $\angle X \equiv \angle Y, \equiv \angle Z$. Verify the fact also by drawing and measuring the angles.

Hence,

Angles in the same segment are equal.

Instead if $\overline{AB}$ subtends angles such that

$$\angle APB \equiv \angle AQB \equiv \angle ARB \equiv \angle ASB$$

at P, Q, R, S respectively, then A, P, Q, R, S, B lie on the arc of a circle.

This can be verified by drawing figures.

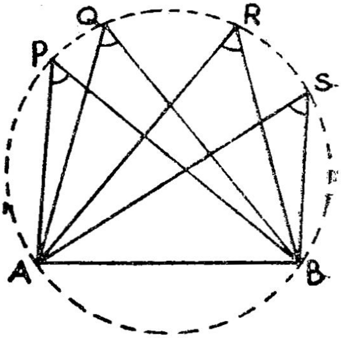


Fig. 7-49.

Exercise 7-2-5

1. In fig. 7-50, $\overline{AB}$ and $\overline{CD}$ are diameters of a circle with center O. If $m\angle OBC = 40°$ find the measures of $\angle A$, $\angle D$, $\angle C$. Verify the fact $\triangle AOD \equiv \triangle COB$ and give reason.

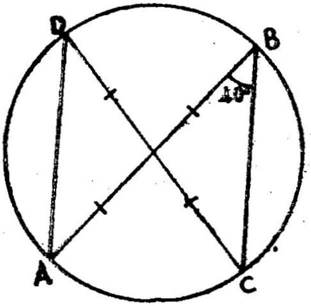


Fig. 7-50.

2. In fig. 7-51, verify $\triangle APD \sim \triangle BPC$.

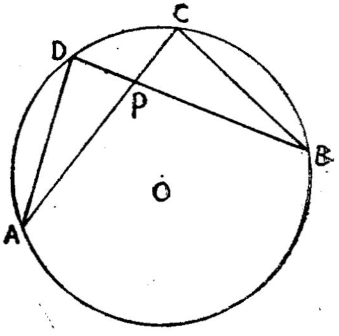


Fig. 7-51.

3. In fig. 7-52, in the half plane APB, if $m\angle APB$ is constant, what is the locus of P ?

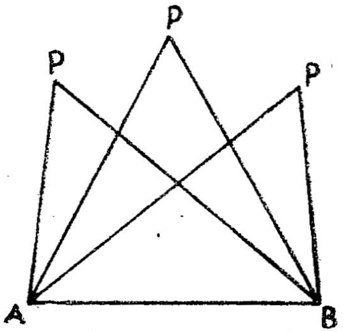


Fig. 7-52.

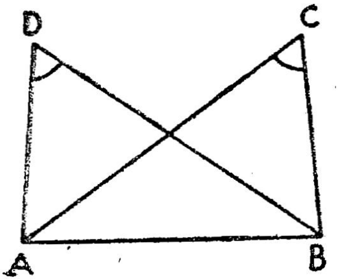


Fig. 7-53.

In fig. 7-53, if $m\angle ADB = m\angle ACB$, what can you say about A, B, C, D?

ANSWRES

(1) $m\angle A = m\angle D = m\angle C = 40°$; Yes.

(2) SAS

(3) $\overset{\frown}{APB}$

(4) The points are concyclic.

7-2.6 Central angles

Fold a circular card board into two equal halves. Again fold the semi-circles into equal halves. We get O,

the centre of the circle. Then fold the circle along any one chord, $\overline{AB}$. Fold along $\overline{AO}$ and $\overline{BO}$ and make the $\angle AOB$ at the centre. In the half plane of the circle where O lies, mark a point P on the circumference. Fold along $\overline{AP}$ and $\overline{BP}$ and make an $\angle APB$. Now cut off the central $\angle AOB$ and fold it into two parts having equal angles. Place one part of the angle on the inscribed $\angle APB$. We see that it is equal to $\angle APB$.

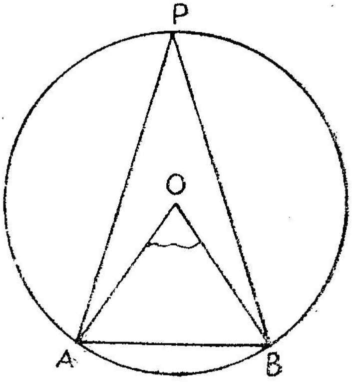


Fig. 7-54.

$$\left[\text{We find } \frac{1}{2}\, m\angle\, AOB = m\angle\, P\right]$$

Hence,

> In a circle, the measure of the central angle is twice the measure of the inscribed angle.

The following examples explain this fact.

1.

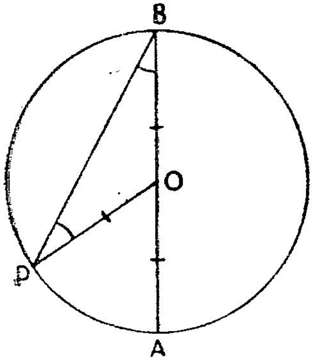


Fig. 7-55.

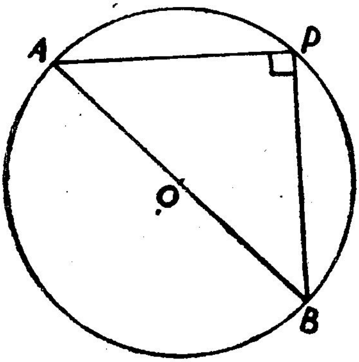
In Fig. 7-55, O is the centre, $\overline{AB}$ is a diameter and $\widehat{AP}$ is an arc of the circle. $\angle AOP$ is the central angle. $\angle ABP$ is the inscribed angle. In $\triangle\ BOP$, $\angle AOP$ is the exterior angle. $m\angle AOP = m\angle OBP + m\angle OPB$. Since $\overline{OB} \equiv \overline{OP} \equiv$ radius of the circle.

$\triangle$ OPB is isosceles

Hence, $m\angle OBP = m\angle OPB$

$\therefore\ m\angle AOP = 2\, m\angle OBP$.

2. In fig. 7-56, O is the centre and, $\overline{AB}$ is a diameter.



$\angle AOB$ is the angle at the centre.

$m\angle AOB = 180°$

$\angle APB$ is an angle in a semicircle.

$\therefore \quad m\angle APB = 90°$

$\therefore \quad m\angle AOB = 2\, m\angle APB.$

Fig. 7-56.

Exercise 7-2·6

1\.

In fig 7-57,
if $m\angle POQ = 120°$,
find $m\angle PRQ$.

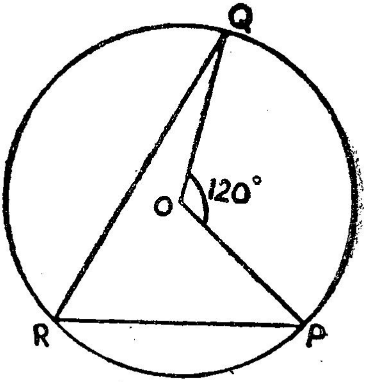


Fig. 7-57.

2\.

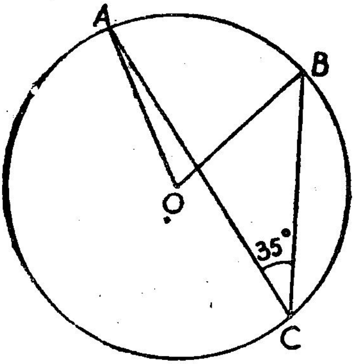


Fig. 7-58.

In fig. 7-58,
if $m\angle ACB = 35°$,
find $m\angle AOB$.

3.

In fig. 7-59, verify the statement $m\angle LOM = m\angle N + m\angle P$

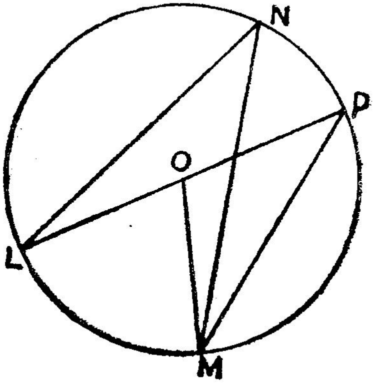


Fig. 7-59.

4.

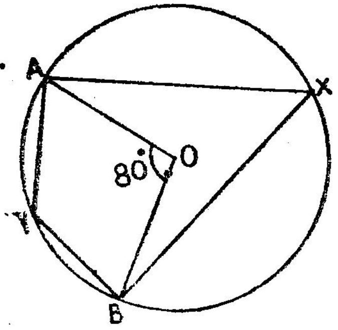


Fig. 7-60.

In fig. 7-60, if $m\angle AOB = 80°$, find $m\angle AYB$.

ANSWERS

(1) 60° (2) 70° (3) True (4) 140°

7-2-7. Tangent

If a line in the plane of a circle intersects the circle in only one point, the line is said to be a tangent to the circle.

Draw a circle with O as the centre. Take any point A on its circumference. Draw a tangent $\overleftrightarrow{AT}$ at A. Using protractor, measure and find $m\angle OAT = 90°$.

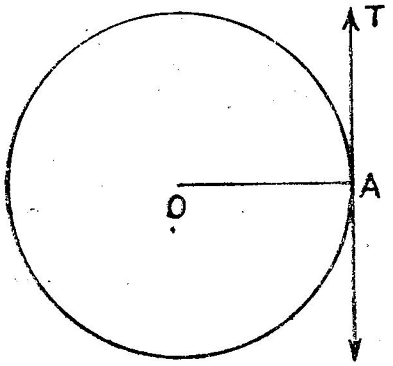


Fig. 7-61.

Hence,

> A tangent to a circle in a plane is perpendicular to the radius drawn at the point of contact.

Draw a radius $\overline{OA}$. The common point between the radius and the circle is A. At A draw a perpendicular $\overleftrightarrow{AT}$. This is the tangent for the cricle.

Hence,

> If a line lying in the plane of a circle is perpendicular to a radius at its outer end, then it is a tangent to the circle at that point.

From the above theorems we learn how to draw tangents to circles.

Exercise 7-2-7

1. How many tangents can be drawn from a point A on the circumference of a circle ?

2. O is the centre of two concentric circles. Chord $\overline{PQ}$ of the outer circle forms the tangent for the inner circle. What can you infer about $\overline{PA}$ and $\overline{AQ}$?

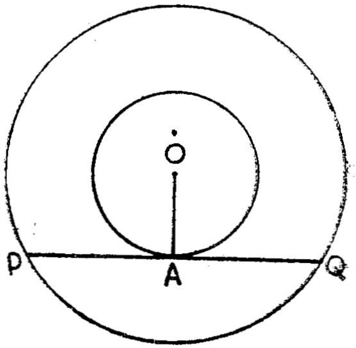


Fig. 7-62.

3. $\overline{AB}$ is a diameter of the circle with centre O. $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$ are tangents to the circles. State the nature of $\overleftrightarrow{AP}$ and $\overleftrightarrow{BQ}$?

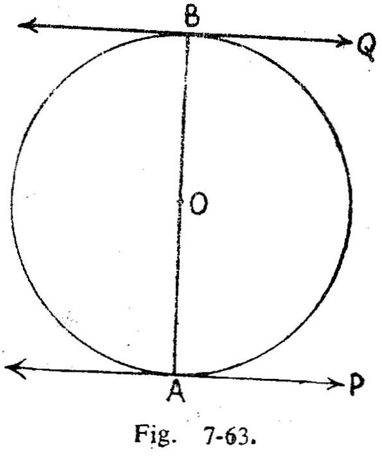


Fig. 7-63.

4. In Fig. 7-64 the length of the radius $\overline{OA}$ is 5 cm. Distance between O and T is 13 cm. Find AT.

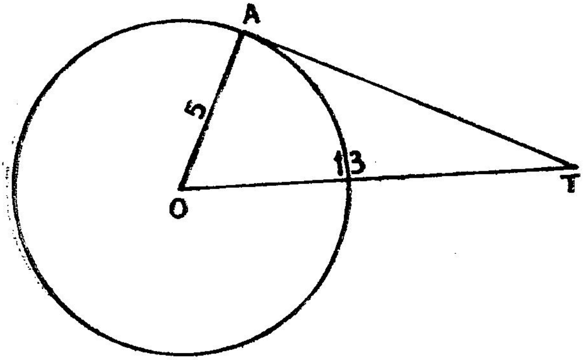


Fig. 7-64.

ANSWERS

(1) One (2) $\overline{PA} \equiv \overline{AQ}$ (3) Parallel lines
(4) 12 cm

7-2.8. Pairs of tangents

Mark a point O on a piece of paper. Draw a circle with O as centre. Mark a point P outside the circle. From P draw tangents $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ to the circle. Measure the length of $\overline{PA}$ and $\overline{PB}$. You will find $m\,\overline{PA} = m\,\overline{PB}$.

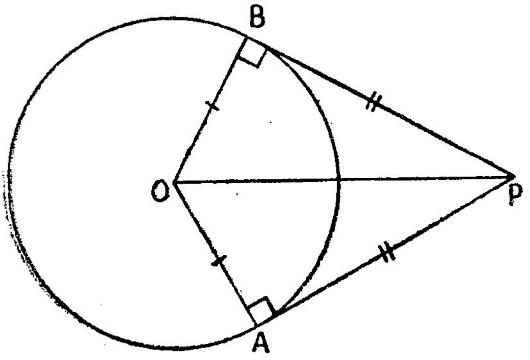


Fig. 7-65.

Cut the quadrilateral $AOBP$ and fold it along $\overline{OP}$. $\overline{OA}$ falls on $\overline{OB}$ and $\overline{PA}$ on $\overline{PB}$. Also $\overline{PA} \equiv \overline{PB}$, $\angle A \cong \angle B$ and $m\angle OAP = m\angle OBP = 90°$.

Note that $m\angle POA = m\angle POB$.

$\overline{PA} \perp \overline{OA}$ $\therefore$ $m\angle PAO = 90°$.

$\overline{PB} \perp \overline{OB}$ $\therefore$ $m\angle PBO = 90°$.

In right triangles OAP and OBP

(i) Hypotenuse $\overline{OP}$ is common.

(ii) Since $\overline{OA}$ and $\overline{OB}$ are radi of the circle, $\overline{OA} \equiv \overline{OB}$

Hence, by the Hypotenuse—Leg Theorem,

$\triangle\ OAP \equiv \triangle\ OBP$

$\therefore$ $\overline{PA} \equiv \overline{PB}$

Also $\angle POA \equiv \angle POB$

Hence,

> The two tangent segments drawn to a circle from an external point are congruent. They subtend equal angles at the centre.

Exercise 7-2·8

1. In Fig. 7-66

   if $OB = r$

   $OP = d$

   (i) Find PB.

   (ii) What type of quadrilateral is $AOBP$?

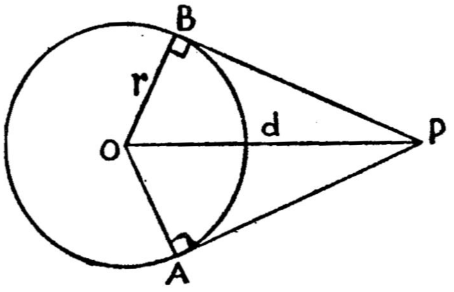


Fig. 7-66.

2. In Fig. 7-67, $\overline{AL}$, $\overline{AM}$ are tangent segments. If $AM = 6$, find AL. If $m\angle OAM = 30°$, find, (i) $m\angle LOM$ and (ii) $m\angle MLA$.

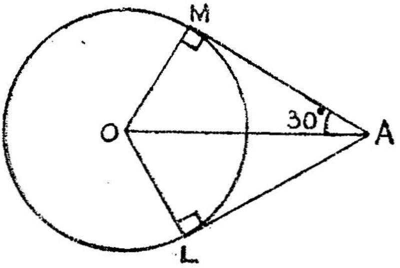


Fig. 7-67.

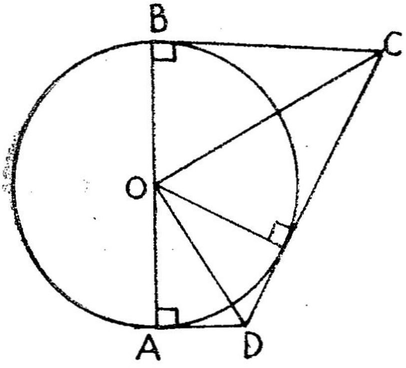


Fig. 7-68.

3. In Fig. 7-68, $\overline{AOB}$ is a diameter. If $\overline{AD}$, $\overline{DC}$, $\overline{CB}$ are tangent segments, find $m\angle DOC$.

4. In a circle tangents drawn at the points A, B, C, D form a quadrilateral $PQRS$. Prove that $PQ+RS=QR+SP$.

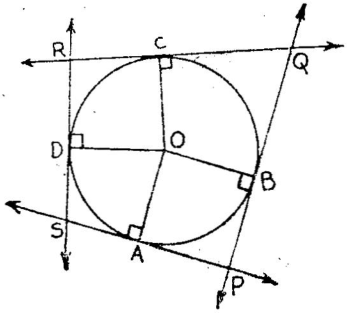


Fig. 7-69.

ANSWERS

(1) $\sqrt{d^2 - r^2}$, cyclic quadrilateral

(2) 6, (i) 120° (ii) 60° (3) 90°

7-2.9. Tangent—Chord angle

From a point A on a circle, draw a tangent $\overleftrightarrow{PAQ}$. From the point of contact A, draw a chord $\overline{AB}$. Draw an $\angle ACB$ in the major segment and $\angle ADB$ in the minor segment. Cut out $\angle ACB$ and place it on $\angle QAB$. Similarly cut out $\angle ADB$ and place it on $\angle BAP$.

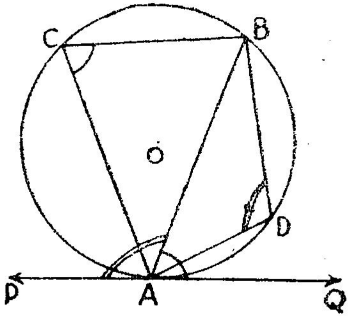


Fig. 7-70

You will find that

(1) $\angle QAB \equiv \angle ACB$

(2) $\angle BAP \equiv \angle ADB$

Draw different figures and verify this statement using protractor.

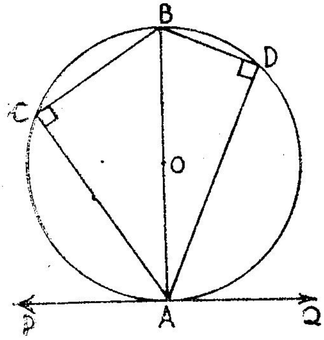


Fig. 7-71

If chord $\overline{AB}$ happens to be the diameter, then $\overline{AB} \perp \overline{PAQ}$.

$m \angle QAB = m \angle BAP = 90°$.

$\therefore$ $ADBC$ is a cyclic quadriateral. $\angle C$ and $\angle D$ are angles in a semi circle.

$\therefore$ $m \angle C = m \angle D = 90°$

$\therefore$ $m \angle QAB = m \angle ACB$

$m \angle BAP = m \angle ADB$

Hence,

> The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

Conversely,

> If the angle between a chord and a line drawn at one end of the chord is equal to the angle in the alternate segment, then the line is a tangent to the circle.

Exercise 7-2-9

1.

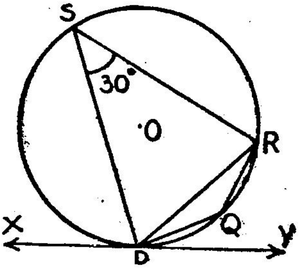


Fig 7-72.

In Fig. 7-72, $\overleftrightarrow{XPY}$ is a tangent.

Find (i) $m\angle RPY$

(ii) $m\angle RPX$ and

(iii) $m\angle Q$.

2. In Fig. 7-73, $\overleftrightarrow{AB}$ is a tangent. If $m\angle BOD = 120°$, find

(i) $m\angle BED$

(ii) $m\angle DBC$.

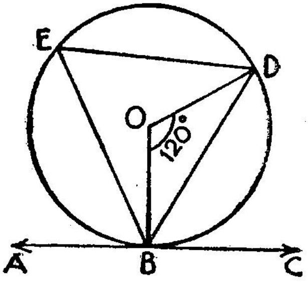


Fig. 7-73.

3.

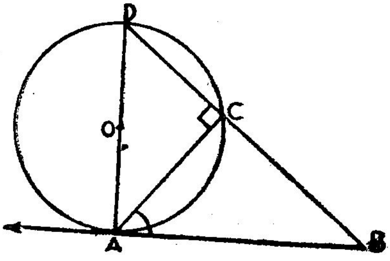


Fig. 7-74

In Fig. 7-74, $\overrightarrow{BA}$ is a tangent ray. If $m\angle D = 50°$ find $m\angle B$.

4. In Fig. 7-75, $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are tangent rays. If $m\angle A = 80°$, find $m\angle D$.

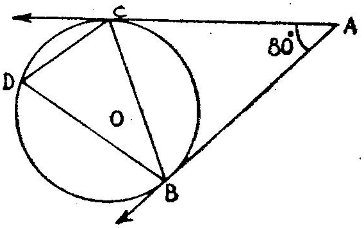


Fig. 7-75.

ANSWERS

(1) (i) 30° (ii) 150° (iii) 150°

(2) (i) 60° (ii) 60° (3) 40° (4) 50°

7-2-10 Intersecting Circles

Draw on a piece of paper two circles with the same radius. Cut out these circles and place one upon the other as shown in Stage 1. in Fig. 7-76.

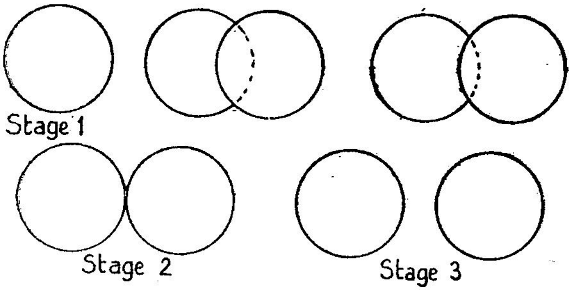


Fig. 7-76

Then slowly move the circles till they touch each other as shown in Stage 2. Move them further. Now they are away from each other as shown in Stage 3.

In between stages (1) and (2), the circles intersect only at 2 points. We can verify it by drawing figures. If the circles touch at three points, they lie on each other. Verify the fact

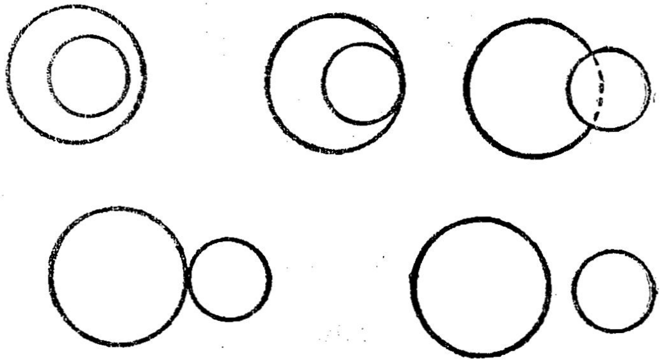


Fig. 7-77

with a few more pairs of circles. Now we see that two circles cannot intersect at 3 points.

From this we learn the following:

Circles with different centres or different radii cannot intersect each other at more than two points.

7-2-11 Touching Circles

We have learnt in 7-2-10 that two circles can touch each other externally or internally at only one point.

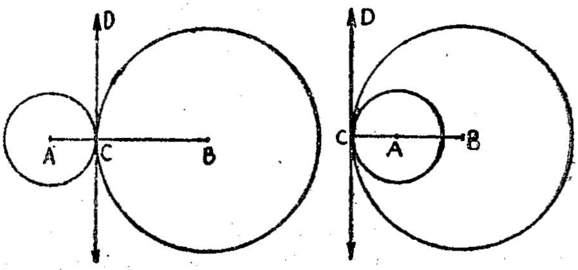


Fig. 7-78

C is the point of contact. The line segment joining A, the centre of the first circle and C is the radius $\overline{AC}$. $\overleftrightarrow{CD}$ is the tangent. $\overleftrightarrow{CD} \perp \overline{AC}$. Hence, $m\angle ACD = 90°$.

Also the line segment joining B, the centre of the second circle and C is the radius BC of the second circle. $\overleftrightarrow{CD}$ is perpendicular to $\overline{BC}$ also. $m\angle BCD = 90°$. So $\overleftrightarrow{CD}$ is a tangent for the second circle also. Hence, the tangent drawn through C becomes a tangent to both the circles. $\overleftrightarrow{CD}$ is called **a common tangent.**

Hence,

> Circles touching each other have a common tangent at the point of contact.

We can verify this statement by drawing more pairs of circles.

7-2-12 Centres of Touching Circles

Let us refer to Fig. 7-78. We have learnt that circles with centres A, B touching externally or internally at the point of contact C have a common tangent $\overleftrightarrow{CD}$; and $\overleftrightarrow{CD}$ is perpendicular to $\overline{AC}$.

$$\overleftrightarrow{CD} \perp \overline{AC} \Longrightarrow m\angle ACD = 90°$$

$$\overleftrightarrow{CD} \perp \overline{BC} \Longrightarrow m\angle BCD = 90°$$

So, $m\angle ACD = m\angle BCD = 90°$

$m\angle ACD + m\angle DCB = 90° + 90° = 180°$

$\overline{ACB}$ is a line segment. Hence, we know that the points, A, C, B are collinear.

Hence,

> If two circles touch each other, their centres and the point of contact are in one straight line.

If the radius of the first circle is r_1, and that of the second is r_2, the distance between the centres.

$d = r_1 + r_2$ (when they touch externally),

$d = r_1 - r_2$ (when they touch internally, $r_1 > r_2$)

Exercise 7-2-12

1. Find the distance between the centres of two circles of radii 5 cm and 7 cm, if they touch externally.
2. What is the distance between the centres of two circles of diameters 10 cm and 12 cm, if they touch internally ?
3. Three circles of radii 5 cm, 6 cm, 7 cm touch externally, each circle touching the other two. Find the sides of the triangle formed by joining their centres.
4. Two circles of diameter 4 cm are drawn within a circle of diameter 8 cm. Find the distance between their centres.

ANSWERS

(1) 12 cm (2) 1 cm (3) 11 cm 13 cm 12 cm

(4) 4 cm

7-2-13. Intersecting Chords

Draw a circle. Draw two intersecting chords $\overline{AB}$ and $\overline{CD}$.

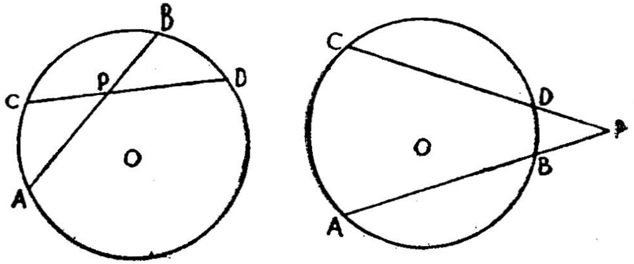


Fig. 7-79

Let them intersect at *P*. *P* may be inside or outside the circle. Measure *PA*, *PB*, *PC*, *PD* and tabulate the results.

No.	*PA*	*PB*	*PC*	*PD*	*PA . PB*	*PC . PD*
1						
2						
3						
...						
...						
...						

From the table, we learn $PA \cdot PB = PC \cdot PD$

Hence,

> If two chords of a circle intersect, then the product of the lengths of the two segments of the one is equal to the product of the lengths of the two segments of the other.

Example 1 :

In Fig. 7-80, in $\triangle APC$ and $\triangle PBD$

$m\angle A = m\angle D$ (Inscribed angles of $\overset{\frown}{BC}$)

$m\angle C = m\angle B$ (Inscribed angles of $\overset{\frown}{AD}$)

$m\angle APC = m\angle DPB$ (Vertical pair)

$\therefore \quad \triangle APC \sim \triangle PDB$

$\therefore \quad \dfrac{PA}{PD} = \dfrac{PC}{PB}$

$PA \cdot PB = PC \cdot PD$.

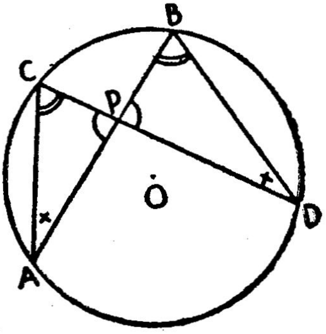


Fig. 7-80.

Example 2 :

In fig. 7-81 in $\triangle APC$, $\triangle DPB$

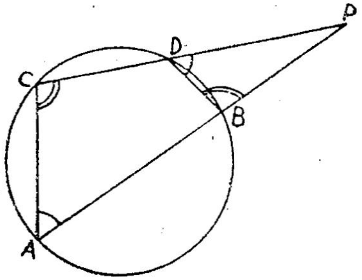


Fig. 7-81.

$m\angle A = m\angle PDB$ (In a cyclic quadrilateral, exterior angle= int. opp. angle)

$m\angle C = m\angle DBP$ (In a cyclic quadrilateral, exterior angle= int. opp. angle.)

$m\angle APC = m\angle DPB$ (Common angle)

$\therefore \quad \triangle APC ||| \triangle DPB$

$$\therefore \quad \frac{PA}{PD} = \frac{PC}{PB}$$

$$PA \,.\, PB = PC \,.\, PD$$

Note :

While naming the segments, start naming them from the point of intersection.

In fig. 7-81, segments are *PA, PB, PC, PD* and not *AB, BP, CD, DP*.

Exercise 7-2-13.

1. In Fig. 7-82 find x.

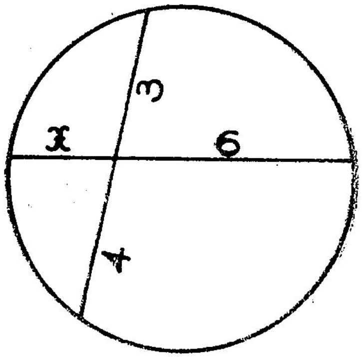


Fig. 7-82

2. In Fig. 7-83, O is the centre of the circle.

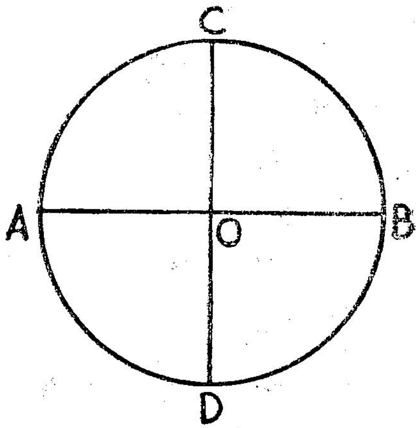
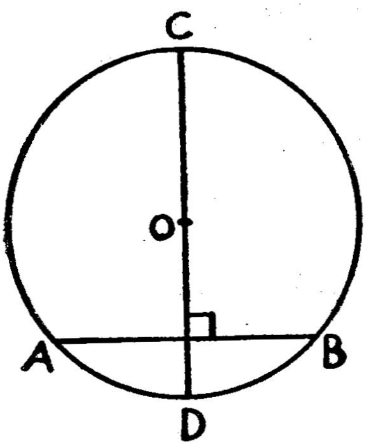


Fig. 7-83

If $OA \,.\, OB + OC \,.\, OD = 32$, find the radius.

3.



In Fig. 7-84, Chord $\overline{AB} \perp$ diameter $\overline{CD}$. The radius is 10 cm. The distance of the chord from the centere is 6 cm. Find the length of the chord.

Fig. 7-84.

4. In Fig. 7-85, find the measure of $\overline{CD}$.

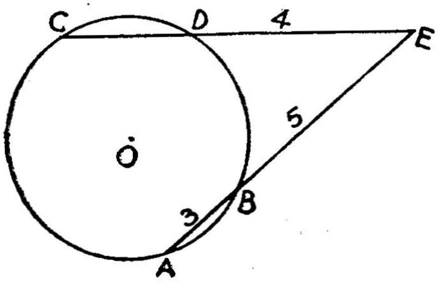


Fig. 7-85

ANSWERS

(1) 2 (2) 4 (3) 16 cm (4) 6

7-2-14. Secants and Tangents

Draw a circle. Mark a point A outside the circle. From A draw a secant $\overline{ABC}$. Let it cut the circle at B and C. From

A draw a tangent $\overline{AD}$. Measure *AC*, *AB*, *AD* and tabulate the results.

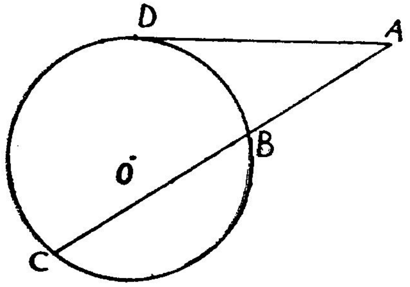


Fig 7-86

For different positions of $\overline{ABC}$, tabulate the measures arrived at.

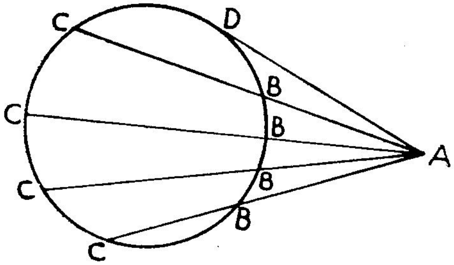


Fig. 7-87

No.	*AC*	*AB*	*AD*	*AC* . *AB*	AD^2
1					
2					
3					
.					
.					
.					

From the table we learn $AC \cdot AB = AD^2$

Hence,

> If, from an external point, a secant and a tangent are drawn to the circle, the area of the rectangle formed by the segments of the secant is equal to the area of the square formed by the tangent segment.

Exercise 7-2-14

1. In Fig. 7-88, $\overleftrightarrow{LP}$ is a tangent. Find LP.

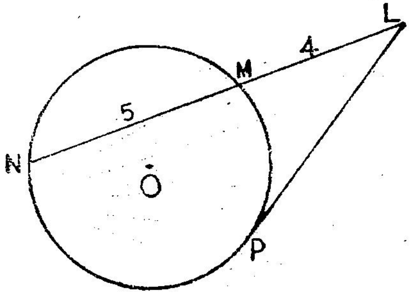


Fig. 7-88

2. In Fig. 7-89 PS is a tangent. Find QR.

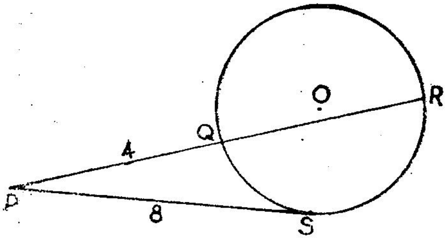


Fig. 7-89

3. In Fig. 7-90, C is the centre of the circle. If $\overleftrightarrow{AE}$ is a tangent, find the radius.

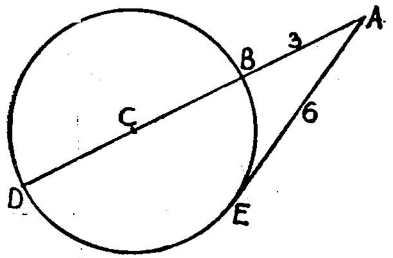


Fig. 7-90

4. In Fig. 7-91, if $\overleftrightarrow{XT}$ is a tangent, find XY.

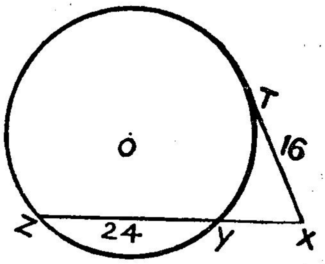


Fig. 7-91

ANSWERS

(1) 6 (2) 12 (3) 4·5 (4) 8

7-3. Analytical Geometry

7-3.1. Distance Between two points.

We have learnt how to mark ordered pairs in $R \times R$. Let us now learn to find the distance between any two points.

1. Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$

To find the distance AB, draw $\overline{AP}$ and $\overline{BQ}$ perpendicular to the X axis. Draw $\overline{AC} \perp \overline{BQ}$.

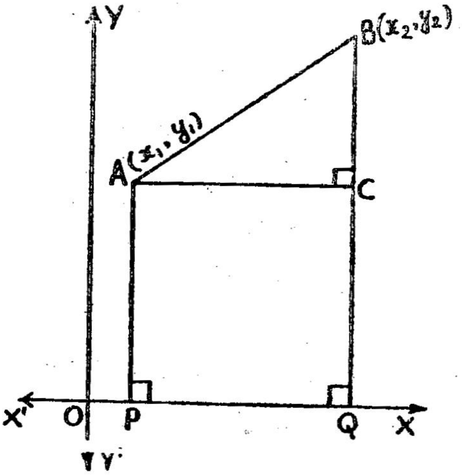


Fig. 7-92

$OP = x_1$, $PA = y_1$; $OQ = x_2$ and $QB = y_2$

$$AC = PQ = OQ - OP$$
$$= x_2 - x_1$$
$$CB = QB - QC$$
$$= QB - PA$$
$$= y_2 - y_1$$

$\triangle$ ABC, $m\angle C = 90°$.

$AB^2 = AC^2 + CB^2$ (Pythagoras Theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$BA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Since $AB = BA$,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2. The distance between the origin O (0, 0) and A (x, y)

$$OA = \sqrt{(x-0)^2 + (y-0)^2}$$
$$= \sqrt{x^2 + y^2}$$

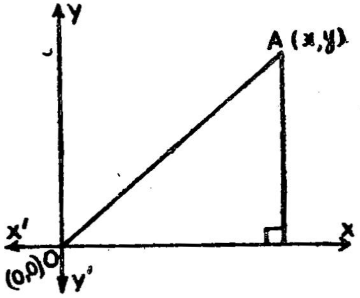


Fig. 7-93

Example 1 :

Find the distance between (5, −3) and (10, 9).

Solution :

The distance between the two points

$$= \sqrt{(10-5)^2 + [9-(-3)]^2}$$
$$= \sqrt{(5)^2 + (12)^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13.$$

Example 2 :

If P (3, 2) and Q (8, 7), find PQ.

Solution :

$$PQ = \sqrt{(8-3)^2 + (7-2)^2}$$
$$= \sqrt{(5)^2 + (5)^2}$$
$$= \sqrt{25 + 25}$$
$$= \sqrt{50}$$
$$= \sqrt{25 \times 2}$$
$$= 5\sqrt{2}$$

Exercise 7-3.1

Find the distance between the following pairs of points.

1. $(3, 7)$, $(-1, 4)$ 2. $(0, -2)$, $(-3, 1)$

3. $(-3, -1)$, $(2, 5)$ 4. $\left(1\frac{1}{2}, 1\frac{3}{4}\right)$, $\left(\frac{1}{2}, \frac{3}{4}\right)$

5. $(m, 0)$, $(n, 0)$ 6. $(0, a)$, $(-b, 0)$

ANSWERS

(1) 5 (2) $3\sqrt{2}$ (3) $\sqrt{61}$ (4) $\sqrt{2}$

(5) $|(m-n)|$ (6) $\sqrt{a^2+b^2}$

7-3.2 Plane Figures

Example 1 :

The vertices of a triangle are (1, 0), (3, 3) and (5, 0). What type of triangle is it ?

Solution :

Let $A\,(1, 0)$, $B\,(3, 3)$ and $C\,(5, 0)$

$$AB = \sqrt{(3-1)^2 + (3-0)^2}$$
$$= \sqrt{(2)^2 + (3)^2}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$
$$BC = \sqrt{(5-3)^2 + (0-3)^2}$$
$$= \sqrt{2^2 + (-3)^2}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$
$$CA = \sqrt{(1-5)^2 + (0-0)^2}$$
$$= \sqrt{(-4)^2}$$
$$= 4$$

Since $AB = BC$, ΔABC is an isosceles triangle.

Example 2 :

Find the perimeter of the quadrilateral whose vertices are (1, 1), (4, 1), (4, 4) and (1, 4).

Solution :

Let $A(1, 1)$, $B(4, 1)$, $C(4, 4)$ and $D(1, 4)$ be the vertices.

$$AB = \sqrt{(4-1)^2 + (1-1)^2} = \sqrt{(3)^2} = \sqrt{9} = 3$$

$$BC = \sqrt{(4-4)^2 + (4-1)^2} = \sqrt{(3)^2} = \sqrt{9} = 3$$

$$CD = \sqrt{(1-4)^2 + (4-4)^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$$

$$DA = \sqrt{(1-1)^2 + (1-4)^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$$

The perimeter of the quadrilateral $ABCD$

$$= AB + BC + CD + DA$$
$$= 3 + 3 + 3 + 3$$
$$= 12$$

Exercise 7-3·2

1. Find the type of the triangle whose vertices are the following :
   (i) (0, 1), (−2, 4), (−4, 1)
   (ii) $(2 + \sqrt{3}, 5)$, (2, 6). (2, 4)

2. Find whether the triangle with the following vertices is a right triangle.
   (i) (3, −2), (1, −2), (1, 2)
   (ii) (−2, −1), (−2, −4), (3, −5)

3. Find the perimeter of the quadrilateral whose vertices are as follows :
   (i) (−2, 1), (−2, 4), (1, 4), (1, 1)
   (ii) (−5, 1), (−5, 3), (−1, 3), (−1, 1)

ANSWERS

1. (i) isosceles triangle (ii) equilateral triangle

2 (i) Yes (ii) No 3. (i) 12 (ii) 12

7.3.3 Equation of a straight line

Let the straight line AB meet the X axis at A and the Y axis at B.

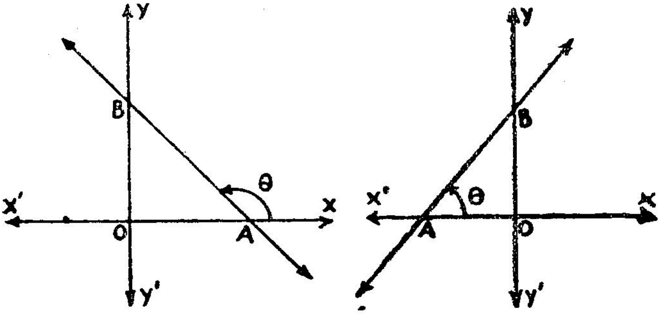


Fig. 7-94.

This line makes an angle $\angle XAB\ (= \theta)$ with the X axis, the tangent of which is equal to a constant, say m.

$\tan \angle XAB = m$. The constant m is also called the slope of the line.

Therefore, the slope of $\overleftrightarrow{AB}$, $m = \tan \theta$. The intercept made by $\overleftrightarrow{AB}$ on the X axis is $\overline{OA}$ and that on the Y axis is $\overline{OB}$.

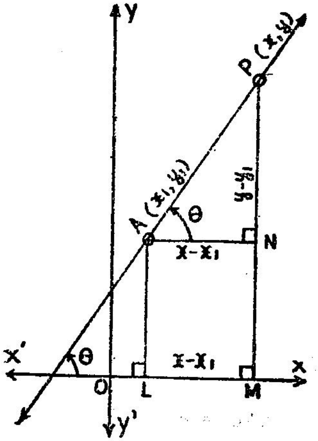


Fig. 7-95.

1. Let m be the slope of the line passing through the point $A(x_1, y_1)$. Let $P(x, y)$ be any point on the line.

Then $m = \dfrac{y - y_1}{x - x_1}$.

$$\therefore \boxed{y - y_1 = m(x - x_1)}$$

which is the equation of the line.

2. Similarly the equation of the line passing through the point $A(o, c)$ is $y - c = m(x - 0)$.

i. e. $\boxed{y = mx + c}$

In the above equation, m is the slope of the line and c is the length of the intercept of the line on the Y axis.

3. Let $P(x, y)$ be any point on the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

The slope of $\overleftrightarrow{AP}$, $m = \dfrac{y - y_1}{x - x_1}$

The slope of $\overleftrightarrow{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Since $\overleftrightarrow{AP}$ and $\overleftrightarrow{AB}$ are one and the same line, their slopes are equal.

$$\therefore \quad m = \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$\therefore$ The equation of $\overleftrightarrow{AB}$ becomes

$$\boxed{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

which can be written as

$$y - y_1 = m(x - x_1); \quad \text{where} \left[m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

Example :

Find the equation of the line passing through the points (2, 3) and (7, 8).

Solution :

Let $x_1 = 2$, $y_1 = 3$, $x_2 = 7$ and $y_2 = 8$

The slope of the line $m = \dfrac{8 - 3}{7 - 2} = \dfrac{5}{5} = 1$

The equation of the line passing through the points (2, 3) and (7, 8) is

$$(y - y_1) = m\,(x - x_1)$$

$$(y - 3) = 1\,(x - 2)$$

$$y = x + 1$$

$$x - y + 1 = 0$$

Exercise 7.3.3

Find the equation of the lines passing through the following pairs of points.

1. (2, 3), (−1, 6)
2. (3, −2), (5, −3)
3. (−3, −6), (−2, −1)
4. $(a, 0)$, $(0, b)$

ANSWERS

1. $x + y - 5 = 0$
2. $x + 2y + 1 = 0$
3. $5x - y + 9 = 0$
4. $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

7.3.4 $y = mx + c$

The equation $y = mx + c$ yields different lines for different values of m and c.

1.

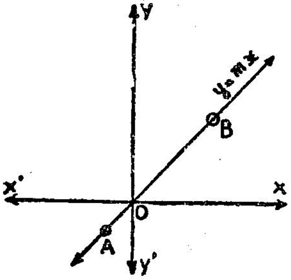


Fig. 7-96

The straight line passing through the origin O (0, 0) does not make any intercept along the Y axis. Therefore $c=0$. Hence, the equation of any straight line passing through the origin is $y = mx$.

2. The angle made with the X axis by a straight line parallel to the X axis is 0. Therefore $m = \tan\theta = 0$. Hence the equation of any straight line parallel to the X axis is $y = c$.

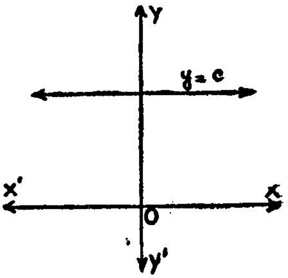


Fig. 7-97

3. The slope of the X axis is 0, and it does not make any intercepet along the Y axis. Hence, the equation of the X axis is $y = 0$.

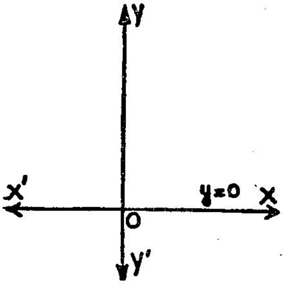


Fig. 7-98

4. The equation of any straight line parallel to the Y axis is $x = c$, where c is the intercept along the X axis.

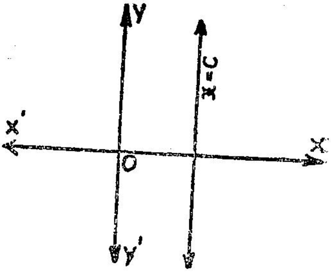


Fig. 7-99

5. The equation of the Y axis is $x = 0$. fig. 7-100

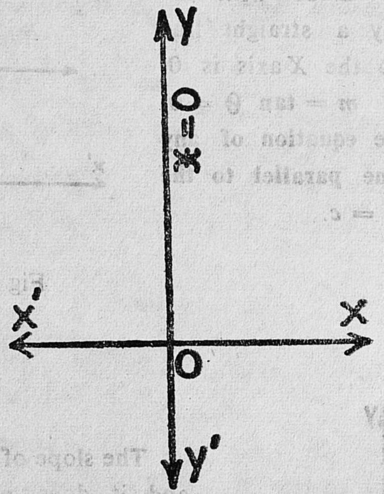


Fig. 7-100

We shall now find the equation of any line.

Example 1 :

What is the equation of the line with slope 5 and passing through the origin ?

Solution :

Slope $m = 5$

The equation of any straight line which passes through the origin 0 (0, 0) is $y = mx$.

Hence the equation of the required straight line is $y = 5x$.

Example 2 :

Find the equation of the line parallel to the X axis and whose Y intercept is -7.

Solution :

The Y intercept $c = -7$

The equation of a line which is parallel to the X axis is $y = c$. Hence, the equation of the required line is $y = -7$.

Exercise 7·3·4

1. Find the equation of the line passing through the origin and with slope : (i) 9 (ii) -5

2. Find the equation of the line which is parallel to the X axis with the following Y intercept : (i) 3 (ii) -4

3. Find the equation of the line which is parallel to the Y axis with the following X intercept : (i) -15 (ii) 10

ANSWERS

1. (i) $y = 9x$ (ii) $y = -5x$
2. (i) $y = 3$ (ii) $y = -4$
3. (i) $x = -15$ (ii) $x = 10$

7.3.5 Finding slopes and intercepts

The equation of a line is $y = mx + c$. We can find the slope and the Y intercept of a line from the above relation.

Example 1 :

Find the slope and intercept on the Y axis of the line $y = 3x - 8$.

Solution :

The equation of the line is $y = 3x - 8$.

Here $m = 3$, and $c = -8$

Hence the slope of the line is 3 and the y intercept is -8.

When the equation of a line is given in the form $ax + by + c = 0$, we transform the equation to the form $y = mx + c$ and then find its slope and the y intercept.

Example 2 :

Find the slope and Y intercept of the line $8x + 5y - 7 = 0$.

Solution :

$$8x + 5y - 7 = 0$$

i.e. $5y = -8x + 7$

$$y = \frac{-8x + 7}{5}$$

$$y = -\frac{8}{5}x + \frac{7}{5}$$

Here $m = -\frac{8}{5}$ and $c = \frac{7}{5}$

Hence the slope is $-\frac{8}{5}$ and the intercept on the Y axis is $\frac{7}{5}$

Exercise 7·3.5

Find the slope and the intercept on the Y axis of the following lines.

(1) $y = 8x - 5$ (2) $y = 3x + 7$ (3) $y = -9x - 3$

(4) $6x + 7y - 3 = 0$ (5) $-5x + 4y - 2 = 0$

(6) $7x - 9y = -1$

ANSWERS

(1) 8, −5 (2) 3, 7 (3) −9, −3

(4) $-\frac{6}{7}, \frac{3}{7}$ (5) $\frac{5}{4}, \frac{1}{2}$ (6) $\frac{7}{9}, \frac{1}{9}$.

7.4 Practical Geometry

7.4.1 Tangents : At a point on the circle

We know that the tangent and the radius drawn through the point of contact make a right angle at the point of contact.

Example :

Draw a circle of radius 3 cm. and draw a tangent to it through any point on it.

Construction :

1. Draw a circle of radius 3 cm having O as centre.
2. Mark a point A on it and draw the radius $\overline{OA}$.
3. Draw $\overleftrightarrow{XAY}$ such that $\overline{AX} \perp \overline{OA}$.

Now, $\overleftrightarrow{XY}$ is the tangent to the circle at the point A.

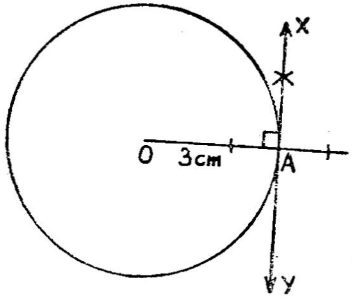


Fig. 7-101

Exercise 7.4.1

1. Draw a circle of radius 5 cm and draw the tangent at any point on it.

2. Draw a circle of diameter 12 cm. Draw a diameter and construct the tangents at the end points of the diameter. How do they appear to be ?

3. Draw a circle of radius 5·5 cm. Draw two diameters at right angles. Construct tangents at each end point of the diameters. What is the enclosed figure ?

4. Draw a circle of radius 4 cm. Draw two diameters intersecting at 50°. Construct tangents at each end point of the diameters. What is the figure enclosed by the tangents ?

ANSWERS

(2) Parallel lines (3) Square (4) Rhombus

7-4-2. Tangents : From a point outside the circle

Example :

Draw a circle of radius 5 cm. Mark a point 13 cm apart from its centre. Construct tangents to the cricle from that point.

Construction :

1. Draw a circle of radius 5 cm having O as centre.

2. Mark the point A such that $\overline{OA} = 13$ cm.

3. Find M, the mid point of $\overline{OA}$.

4. Having M as centre and MO as radius, draw a circle. This cuts the circle with centre O, at two points. Call them X and Y.

5. Join A, X and A, Y.

Now, $\overleftrightarrow{AX}$ and $\overleftrightarrow{AY}$ are the required tangents to the given circle.

We can find that $\overline{AX}$ and $\overline{AY}$ are congruent. This can be verified using Pythagoras Theorem.

$$OA^2 = AX^2 + XO^2$$

$$13^2 = AX^2 + 5^2$$

$$\therefore \quad AX^2 = 13^2 - 5^2 = 169 - 25 = 144$$

$$\therefore \quad AX = \sqrt{144} = 12 \text{ cm}$$

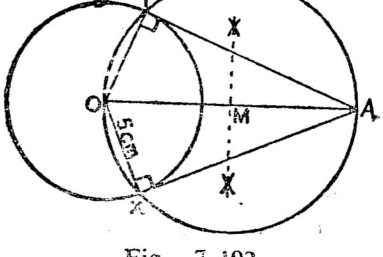


Fig. 7-102.

Exercise 7·4·2

1. Draw a circle of radius 3 cm and mark a point 8 cm away from the centre. Construct two tangents to the circle from that point.

2. Draw a circle of radius 4 cm. Mark a point 9 cm away from its centre and construct the tangents from it. Measure the tangent segments.

3. Draw a circle with 6 cm as diameter and mark a point 5 cm away from the centre Construct the tangents from that point. Measure the lengths of the tangent segments. Verify your answer.

4. Draw a circle of diameter 12 cm and mark a point at a distance of 10 cm from the centre. Construct two tangents. Measure the lengths of the tangent segments. Verify your answer.

7·4·3 Contruction of Triangle I

We have learnt in the Ninth Standard, how to construct triangles according to a given set of measurements. Let us now learn some special constructions.

Here we use the following Theorems,

1. The angle subtended by any arc at the centre is twice the angle subtended by it in the remaining part of the segment.

2. In isosceles triangles, the sides opposite to the equal angles are equal.

To construct a triangle given (i) the base (ii) the vertical angle and (iii) the perpendicular to the base drawn from the vertex.

Example :

Construct a triangle with the following measurements.

$AB = 4$ cm, $m\angle C = 60°$ and the length of perpendicular from C to $\overline{AB}$ is 3 cm.

Construction :

1. Draw a rough figure and mark the given measurements. $m\angle ACB = 60°$; $\therefore$ $m\angle AOB = 2 \times 60° = 120°$ Therefore in the isosceles $\triangle OAB$, $m\angle BAO = 30°$.

2. Draw $\overline{AB}$ such that $AB = 4$ cm.

3. Draw $\overline{AO}$ such that $m\angle BAP = 30°$

4. Drop the perpendicular bisector of $\overline{AB}$ and let it meet $\overline{AP}$ at O.

5. Draw $\overset{\frown}{AB}$ with O as centre and OA as radius.

6. Draw an arc with Q as centre and with a radius of 3 cm. Let i tcut $\overline{QO}$ at R.

7. Through R, draw $\overleftrightarrow{CR}$ perpendicular to $\overleftrightarrow{QR}$ Let it cut $\overline{AB}$ at two points C and C_1.

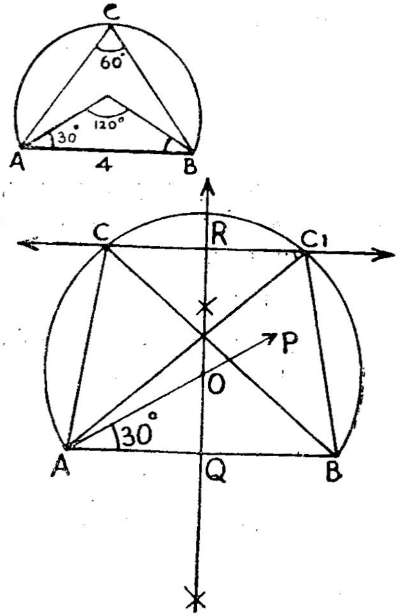


Fig. 7-103.

8. Draw $\overline{AC}$, $\overline{BC}$, $\overline{AC_1}$, and $\overline{BC_1}$.

Now we get two triangles $\triangle ABC$ and $\triangle ABC_1$ for the same set of measurements.

Exercise 7.4-3

1. Draw a triangle with base 6 cm, vertical angle 70° and perpendicular distance from the vertex to the base 4·5 cm. Find its area.

2. Draw $\triangle ABC$ given that $AB = 4$ cm, $m\angle C = 50°$, $\overline{CD} \perp \overline{AB}$ and $CD = 4$ cm. Measure the lengths of the other two sides.

3. The base of a triangle is 5 cm, the vertical angle is 65° and the perpendicular distance from the vertex to the base is 3·5 cm. Construct the triangle and find the measure of the other two angles.

4. Draw a triangle, given $XY = 4·5$ cm, $m\angle Z = 80°$, $\overline{ZA} \perp \overline{XY}$, and $ZA = 4·2$ cm. Find the length of the median through Z.

7.4.4 Construction of triangles II

To construct a triangle given (i) the base (ii) the vertical angle and (iii) the median to the base.

Example :

Draw $\triangle ABC$ given $AB = 5$ cm, $m\angle C = 45°$ and the length of the median from C to $\overline{AB}$ is 4 cm.

Construction :

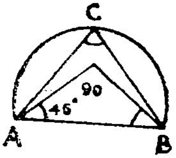
1. Draw a rough figure and mark the given measurements.

2. Draw $\overline{AB}$ with $AB = 5$ cm.

3. $\overset{\frown}{AB}$ such that the inscribed angle in it is 45°.

4. Draw an arc of a circle with centre D, the mid point of $\overline{AB}$ and radius 4 cm. It cuts $\overset{\frown}{AB}$ at two points C, C_1.

5. Draw $\overline{AC}$, $\overline{BC}$, $\overline{AC_1}$, $\overline{BC_1}$
Now we get $\triangle ABC$ and $\triangle ABC_1$ for the given set of measurements.



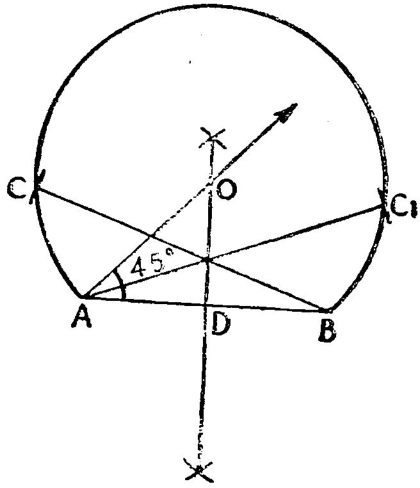


Fig. 7-104.

Exercise 7.4.4

1. Construct a triangle with base 7 cm, vertical angle 60° and the measure of the median from the vertex opposite to the base 6 cm.

2. Construct a triangle with base 5·5 cm, vertical angle 50° and the measure of the median from the vertex opposits to the base 4·5 cm. Measure the other two sides.

3. Draw $\triangle XYZ$ with $XY = 6{\cdot}2$ cm. $m\angle Z = 80°$ and the measure of the median from Z to $\overline{XY}$ is 4·6 cm. Find the measure of the other two angles.

4. The vertical angle opposite to the base 8·5 cm long is 120°. The measure of the median from the vertex opposite to the base is 2 cm. Draw the triangle and find the measurement of the remaining sides and angles.

Note :

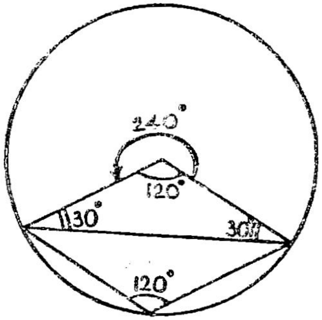


Fig. 7-105.

7-4-5. Enlargement : Triangles

We have learnt that if pairs of corresponding angles are congruent, then the ratios of the corresponding side are equal. Using this theorem we shall study the method of enlarging triangles.

Example :

Construct a triangle with 6 cm, 4 cm, 5 cm as sides. Enlarge the triangle with sides twice as that of the given triangle

Construction :

1. Construct $\triangle ABC$ with $AB = 6$ cm, $BC = 4$ cm, $CA = 5$ cm.
2. Draw $\overrightarrow{AB}$, $\overrightarrow{AC}$ with A as centre of similitude
3. On $\overrightarrow{AB}$ mark a point Q such that $AQ = 12$ cm.
4. On $\overrightarrow{AC}$ mark a point R such that $AR = 10$ cm.

5. Draw $\overline{QR}$.

Now the sides of $\triangle AQR$ is twice that of $\triangle ABC$.

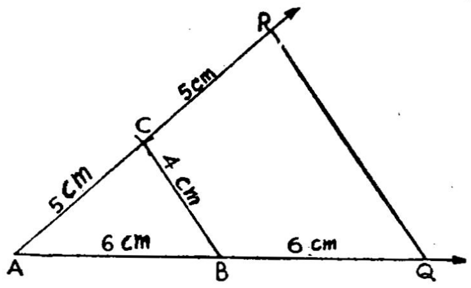


Fig. 7-106.

$QR = 2 \times BC = 2 \times 4 = 8$ cm

Verify $\overline{QR} \parallel \overline{BC}$.

Exercise 7-4·5

1. Construct an equilateral triangle of side 3 cm. Enlarge it such that the enlarged sides are three times the sides of the given triangle.

2. Construct an isosceles triangle with sides 5, 5, 4 units. Enlarge the triangle with the corresponding sides in the ratio 1 : 2.

3. Construct a right triangle with sides 3 cm, 4 cm, 5 cm. Enlarge it such that the area of the enlarged triangle is 4 times that of the enlarged triangle.

4. Construct a triangle with sides 2, 3, 4 units. Enlarge it so that the ratio of the area of the given triangle to that of times that of the enlarged triangle is 1 : 9.

7.4.6 Enlargement : Polygons

Example :

Construct a quadrilateral $ABCD$ with the following measurements. $AB = 4$ cm, $BC = 3{\cdot}5$ cm, $CD = 1{\cdot}5$ cm, $DA = 3{\cdot}5$ cm, $\angle A = 50°$. Enlarge this quadrilateral such that the sides of the given quadrilateral and the corresponding sides of the enlarged quadrilateral are in the ratio 1 : 2.

Construction :

1. Draw a rough figure and mark the given measurements.

2. Construct a quadrilateral $ABCD$ with $AB = 4$ cm, $m \angle A = 50°$, $AD = 3{\cdot}5$ cm, $BC = 2{\cdot}5$ cm, $DC = 1{\cdot}5$ cm.

3. Draw $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ with A as centre of similitude.

4. On $\overrightarrow{AB}$ mark a point Q such that $AB = BQ$.

5. On $\overrightarrow{AC}$ mark a point R such that $AC = CR$.

6. On $\overrightarrow{AD}$ mark a point S such that $AD = DS$.

7. Draw $\overline{QR}$, $\overline{RS}$

Now the sides of the quadrilateral $AQRS$ are twice that of the quadrilateral $ABCD$.

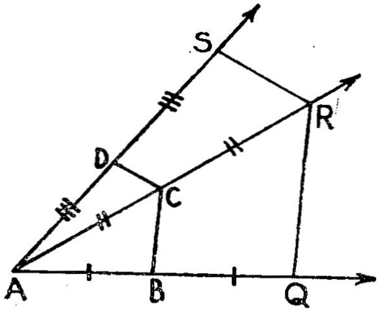


Fig. 7-107.

$AQ = 2\ AB = 2 \times 4 = 8$ cm

$AS = 2\ AD = 2 \times 3{\cdot}5 = 7$ cm

$QR = 2\ BC = 2 \times 2{\cdot}5 = 5$ cm

$RS = 2\ CD = 2 \times 1{\cdot}5 = 3$ cm

Verify $\overline{QR} \parallel \overline{BC}$, $\overline{RS} \parallel \overline{CD}$.

Exercise 7·4·6

Construct a quadrilateral with the following measurements and enlarge it in the given ratio.

1. A square with side 3 cm. Enlargement factor is 3.

2. A rectangle with length 4 cm and breadth 3 cm. The ratio of the sides of this rectangle and the enlarged rectangle is 1 : 2.

3. A parallelogram with sides 5 cm. The measure of one angle is 60°. The area of the enlarged parallelogram is 4 times that of the given parallelogram.

4. In a quadrilateral $PQ = 3$ cm, $QR = 4$ cm, $RS = 5$ cm, $SP = 6$ cm, $RP = 5$ cm. The ratio of the areas of this quadrilateral and the enlarged quadrilateral is 1 : 4.

7.4-7 Construction of the Circumcircle of a triangle

Recall :

In a triangle the perpendicular bisectors of the sides pass through a point. This point is called the **circumcentre** of the triangle.

Example :

Construct the circumcircle of a triangle with sides 8 cm, 5 cm, 7 cm and find its radius.

Construction :

1. Construct $\triangle ABC$ such that $AB = 8$ cm, $BC = 5$ cm, $AC = 7$ cm.

2. Draw the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$. Let them intersect at S.

3. Draw a circle with S as centre and SA as radius. This circle passes through B and C.

This is the circumcircle of $\triangle ABC$.

Verify $SA = SB = SC$.

The radius of the circumcircle $= SA = 4$ cm.

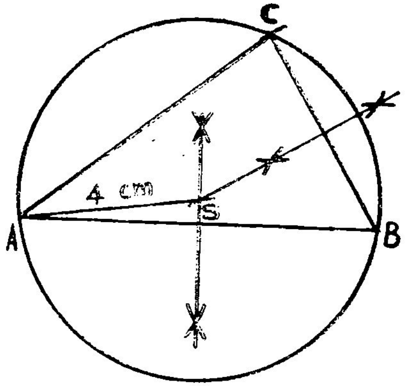


Fig. 7-108.

Note :

The circumcentre lies

(i) inside an acute triangle,

(ii) at the mid point of the hypotenuse of a right triangle,

(iii) outside an obtuse triangle.

Exercise 7-4.7

Draw the circumcircle of the triangles with the following measurements and find the radius.

1. The sides are 4, 5, 6 units.
2. In $\triangle ABC$, $AB = 7$ cm, $BC = 6$ cm, $m\angle B = 45°$.
3. In $\triangle PQR$, $PQ = 5$ cm, $PR = 4{\cdot}5$ cm, $m\angle P = 100°$.
4. In $\triangle XYZ$, $XY = 8$ cm, $XZ = 6$ cm, $m\angle X = 90°$.

7-4·8 Construction of the Incircle of a triangle

Recall :

"The bisectors of the angles of a triangle meet at a point." This point is called the incentre.

Example :

Construct a triangle with sides 6 cm, 7 cm, 8 cm. Construct the incircle and find its radius.

Construction :

1. Construct $\triangle ABC$ with $AB=8$ cm $BC=7$ cm, $AC=6$ cm.

2. Draw the bisectors of $\angle A$ and $\angle B$. Let them intersect at I.

3. Draw $\overline{ID} \perp \overline{AB}$.

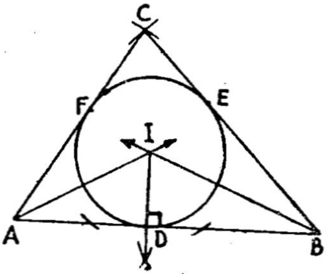


Fig. 7-109.

4. Draw a circle with I as centre and ID as radius. This circle touches $\overline{BC}$ at E and $\overline{AC}$ at F. This is the incircle of $\triangle ABC$.

The radius of the incircle $= ID = 2$ cm

Verify : (i) $ID = IE = IF$.

(ii) $AD = AF$, $BD = BE$, $CE = CF$

Exercise 7-4-8

Draw the incircle of the triangles with the following measurements. Find the radius.

1. Sides 5, 7, 9 units.
2. In $\triangle ABC$, $AB = 10$ cm, $m\angle A = 40°$, $m\angle B = 50°$.
3. In $\triangle PQR$, $PR = 12$ cm, $m\angle P = m\angle R = 40°$.
4. In $\triangle XYZ$, $XY = YZ = ZX = 9$ cm.

REVIEW

1. Conditions for two triangles to be congruent: (a) Side - Angle - Side (b) Angle - side - Angle (c) Side - Side - Side (d) Hypotenuse - Leg (in right triangles).

2. The point of concurrence of the altitudes of a triangle is called the **Orthocentre.**

The point of concurrence of the medians of a triangle is called the **Centroid.**

The point of concurrence of the bisectors of the angle is called the **Incentre.**

The point of concurrence of the perpendicular bisectors of the sides is called the **Circumcentre.**

3. **Circles**

(a) (i) Any two congruent chords of a circle are equidistant from the centre.

(ii) Chords which are equidistant from the centre of a circle are congr ent.

(iii) Congruent chords intercept equal arcs.

(iv) Congruent arcs intercept equal chords.

(b) (i) Angle in a semi circle is a right angle.

(ii) The opposite angles of a cyclic quadrilateral are supplementary.

(c) (i) Angles inscribed in the same arc of a circle are congruent.

(ii) Angle subtended at the centre is twice the angle subtended at the circumference.

(d) (i) The tangent and the radius through the point of contact are perpendicular to each other.

(ii) The two tangent segments to a circle from a point outside the circle are congruent and the angles subtended by these segments at the centre are congruent.

(iii) The angle between a tangent and a chord through the point of contact is congruent to the angle subtended by the chord in the alternate segment.

(e) When two circles touch each other, the point of contact and the centres of the circles lie in the same line.

(f) If two chords of a circle intersect, then the product of the lengths of the two segments of the one is equal to the product of the lengths of the two segments of the other.

4. If pairs of corresponding angles of triangles are congruent then the triangles are similar. In similar triangles the ratios of the corresponding sides are equal.

If the ratio of the sides of similar triangles is $l : m$, then the ratio of their areas is $l^2 . m^2$.

5. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

6. (a) The distance between two points (x_1, y_1) and (x_2, y_2) is $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(b) (i) The slope of the line joining (x_1, y_1) and (x_2, y_2) is $= \dfrac{y_2 - y_1}{x_2 - x_1}$

(ii) The equation of the line joining the two points (x_1, y_1), (x_2, y_2) is,

$$(y - y_1) = m (x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

(c) (i) The equation of a straight line through the origin is $y = mx$.

(ii) The equation of a line parallel to the X axis is $y = c$.

(iii) The equation of a line parallel to the Y axis is $x = c$.

(d) (i) The slope of the line given by $y = mx + c$ is m and the intercept on the Y axis is c.

(ii) The slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$ and the intercept on the Y axis is $-\dfrac{c}{b}$.

SELF-ASSESSMENT TEST I

1. In $\triangle ABC$, if $\overline{AD} \perp BC$, show that $AB^2 - BD^2 = AC^2 - CD^2$.

2. In fig. 7-110 if $m\angle OCB = 40°$, find $m\angle ADC$ and $m\angle AOC$.

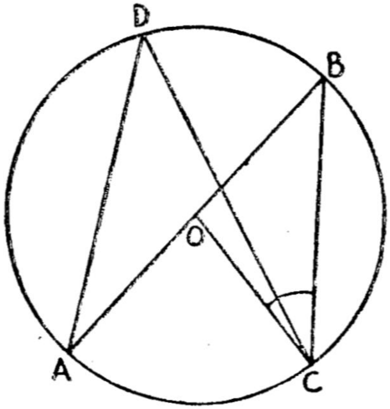


Fig. 7-110.

3. In two concentric circles with radii 12 cm, 13 cm, the chord of one circle is a tangent to the other. Find the length of the chord.

4. 3 circular coins of diameter 2·5 cm each are placed in such a way that each coin touches the other two. Find the side of the triangle formed by joining the points of contact.

5. (3, 3), (3, 6), (6, 3) are the co-ordinates of the vertices of a triangle. Find what type of triangle it is. Also calculate its perimeter.

6. The slope of a straight line is 6. Its intercept on the Y axis is -3 units Find the equation of the line. Also find the equation of the line drawn parallel to the above and passing through the origin.

ANSWERS

(2) 40°, 80° (3) 10 cm (4) 1·25 cm, equilateral triangle (5) Right triangle, $6 + 3\sqrt{2}$ (6) $y = 6x - 3$; $y = 6x$.

SELF - ASSESSMENT TEST II

1. Draw tangents to a circle of diameter 13 cm from a point 11 cm from the centre. Measure the tangent segments.

2. The angle opposite the base of 6.5 cm is 55°. The median from that vertex to that base is 5 cm. Draw the triangle and find the measures of the other two sides.

3. A rhombus of side 4 cm has one of its diagonals equal to 6 cm. Enlarge the rhombus so that each side is doubled.

4. A triangle has two sides equal to 6 5 cm and 8 cm and the inculded angle equal to 110°. Draw its circumcircle and find its radius.

NOT FOR EXAMINATION

MATHEMATICS CLUB — ACTIVITY 6

One Way Traffic

We know Euler as a great mathematician. Once a man asked him, " Is it possible to see the entire Koenigsburg town by crossing the seven bridges there, without passing through any bridge for the second time ?"

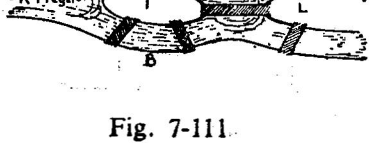


Fig. 7-111

Euler converted this into a mathematical model and tried to find a solution.

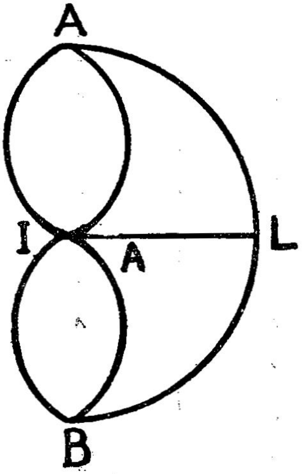


Fig. 7-112.

Let us try to understand the puzzle in his way. Examine the following mathematical figures.

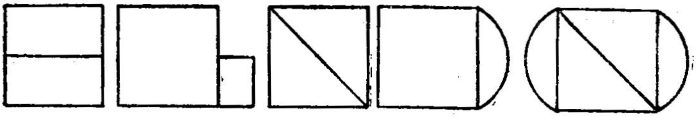


Fig. 7-113.

How many straight lines meet at each vertex in each of the figures ? To test whether it is possible to form one way traffic, let us add one more straight line in each of the figures. Now examine whether it is possible to form the one way traffic.

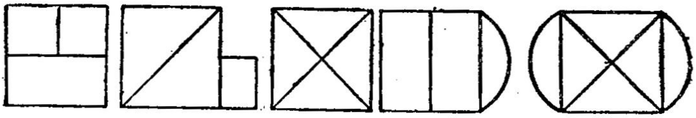


Fig. 7-114.

It is not possible in the first four figures. But it is possible in the last figure.

Figure	Odd vertices	Even vertices	One way Path
1	2	4	Yes
2	2	5	Yes
3	2	2	Yes
4	2	2	Yes
5	2	2	Yes
6	4	4	No
7	4	3	No
8	4	1	No
9	4	2	No
10	0	5	Yes

From this Euler came to the following conclusion.

"It is not possible to form one way path in figures which have more than 2 odd vertices".

Using this theory, we can solve the Koenigsburg riddle.

8. GRAPHS

8.1.1 Graph : $y = x^2$

To draw the graph of a quadratic function of the from $y = x^2$; $x, y \in R$, generally give x the values 0, $\pm$ 1, $\pm$ 2, $\pm$ 3, $\pm$ 4, ... and find the corresponding values of y.

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$	16	9	4	1	0	1	4	9	16

Now, we can plot the ordered pairs .. , $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$ ... on a graph sheet $(R \times R)$ taking proper scales on the axes.

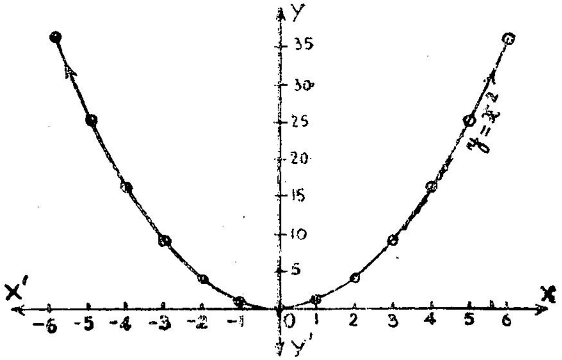


Fig. 8-1.

eg., On the X axis 1 cm → 1 unit.

On the Y axis 1 cm → 5 units.

Draw a smooth curve (without zig-zag) passing through all the points as shown in the figure. This is the graph of the equation $y = x^2$.

Note :

1. The graph lies in the first two quadrants.
2. Tbe graph passes through the origin O (0, 0)
3. The X axis is the tangent to the curve at O.
4. The graph is symmetrical with respect to the Y axis.

Exercise 8-1·1

Draw the graph of $y = x^2$ to different scales.

8.1.2. Graph : $y = x^2 + a$

Example 1 :

Draw the graph of $y = x^2 + 4$; $x, y \in R$

Solution :

x	-4	-3	-2	-1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
4	4	4	4	4	4	4	4	4	4
$y = x^2 + 4$	20	13	8	5	4	5	8	13	20

Scale :

On the X axis 1 cm $\rightarrow$ 1 nuit.

On the Y axis 1 cm $\rightarrow$ 5 units.

The graph of $y = x^2 + 4$ lies in the first two quadrants. It does not pass through the origin O (0, 0). It is symmetrical with respect to the Y axis.

If $x = 0$, then $y = 4$; 4 is the least value for the graph $y=x^2+4$. The graph does not cut the X axis as $y=x^2+4 \neq 0$ for any $x \in R$.

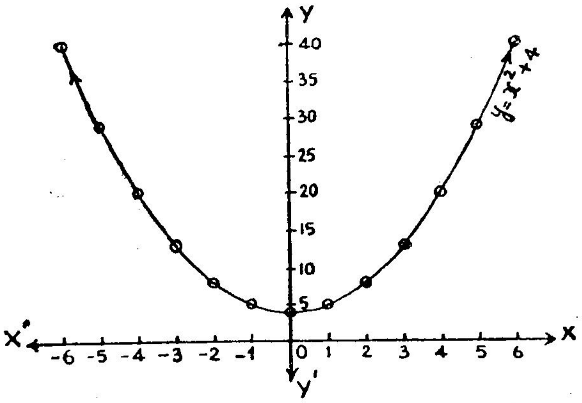


Fig. 8-2.

Example 2 :

Draw the graph of a second degree function, $y = x^2 - 4$: $x, y \in R$.

Solution :

x	-4	-3	-2	-1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
$y = x^2 - 4$	12	5	0	-3	-4	-3	0	5	12

Scale :

On the X axis 1 cm → 1 unit.

On the Y axis 1 cm → 2 units.

The graph of $y = x^2 - 4$ pases through all the four quadrants. It does not pass through the origin O (0, 0). It is symmetrical with respect to the Y axis. The graph cuts the X axis at two points whose co-ordinates are (—2, 0), (2, 0).

When x takes the value $\pm$ 2, the value is 0.

Hence, $y = x^2 - 4 = 0 \Longrightarrow x = -2$ or $+2$.

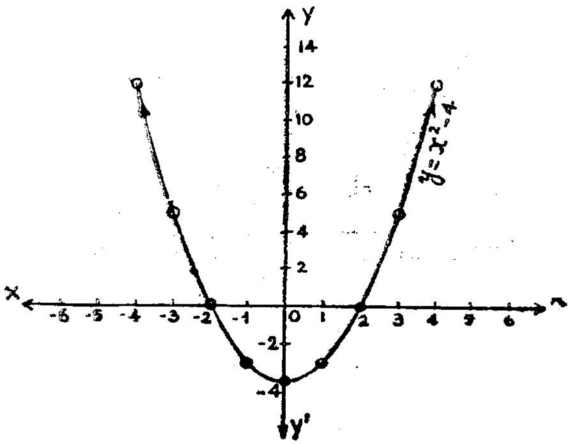


Fig. 8-3.

Exercise 8-1-2

Draw the graphs of the following second degree binomial functions.

(1) $y = x^2 - 1$ (2) $y = x^2 + 9$ (3) $y = x^2 - 16$

DRAW AND LEARN

Draw the graphs of the following to the same scale on the same graph sheet. Study these graphs and know about their shapes.

(1) $y = x^2 + 1$; $y = x^2 + 4$; $y = x^2 + 8$

(2) $y = x^2 - 2$; $y = x^2 - 5$; $y = x^2 - 9$

8-1.3 Graph : $y = x^2 + ax$

Example 1 :

Draw the graph of $y = x^2 + 4x$, $x, y \in R$.

Solution :

x	-6	-5	-4	-3	-2	-1	0	1	2
x^2	36	25	16	9	4	1	0	1	4
$4x$	-24	-20	-16	-12	-8	-4	0	4	8
$y=x^2+4x$	12	5	0	-3	-4	-3	0	5	12

Scale :

On the X axis 1 cm → 1 unit.

On the Y axis 1 cm → 5 units.

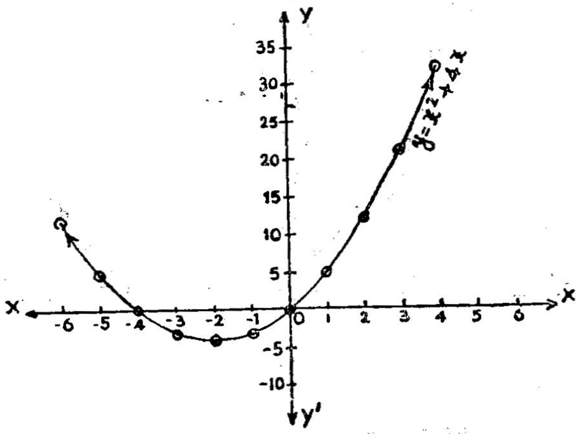


Fig. 8-4.

When x takes the values 0 and -4, y becomes 0.

Hence, $y = x^2 + 4x = 0 \Longrightarrow x = 0$ or -4.

Example 2 :

Draw the graph of the equation

$y = x^2 - 4x,\ x, y \in R$

Solution :

x	-2	-1	0	1	2	3	4	5	6
x^2	4	1	0	1	4	9	16	25	36
$-4x$	8	4	0	-4	-8	-12	-16	-20	-42
$y=x^2-4x$	12	5	0	-3	-4	-3	0	5	12

Scale :

On the X axis 1 cm → 1 unit.

On the Y axis 1 cm → 5 units.

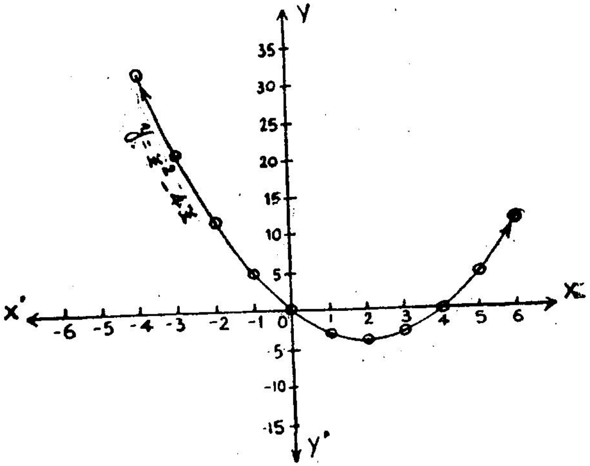


Fig. 8-5.

When x takes the volume 0 and 4, y becomes 0.

Hence, $y = x^2 - 4x = 0 \implies x = 0$ or 4.

Exercise 8-1.3

Draw the graphs of the following equations.

(1) $y = x^2 - 3x$ (2) $y = x^2 + 7x$

(3) $y = x^2 - 16x$ (4) $y = x^2 - 25x$

DRAW AND LEARN

Draw the graphs of the following equations on the same graph sheet taking the same scales on the axes.

(1) $y = x^2 + 2x$; $y = x^2 + 5x$: $y = x^2 + 8x$

(2) $y = x^2 - 2x$; $y = x^2 - 5x$; $y = x^2 - 8x$

8-1-4. Graph : $y = x^2 + ax + b$

Example 1 :

Draw the graph of the following trinomial function $y = x^2 + x + 5$; $x, y \in R$.

Solution :

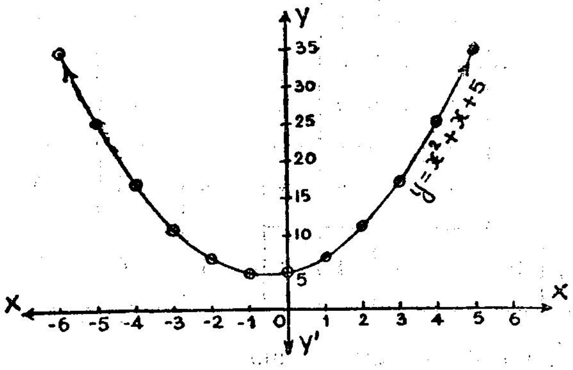


Fig. 8-6.

x	−4	−3	−2	−1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
x	−4	−3	−2	−1	0	1	2	3
5	5	5	5	5	5	5	5	5
$y=x^2+x+5$	17	11	7	5	5	7	11	17

Scale :

On the X axis 1 cm $\rightarrow$ 1 unit.

On the Y axis 1 cm $\rightarrow$ 5 units.

$y \neq 0$ for any value of x. Hence the graph does not cut the X axis.

Example 2 :

Draw the graph of the trinomial quadratic function $y = x^2 - 4x + 4$.

Solution :

x	-2	-1	0	1	2	3	4	5	6
x^2	4	1	0	1	4	9	16	25	36
$-4x$	8	4	0	-4	-8	-12	-16	-20	-24
4	4	4	4	4	4	4	4	4	4
$y = x^2 - 4x + 4$	16	9	4	1	0	1	4	9	16

Scale :

On the X axis 1 cm $\rightarrow$ 1 unit.

On the Y axis 1 cm $\rightarrow$ 5 units.

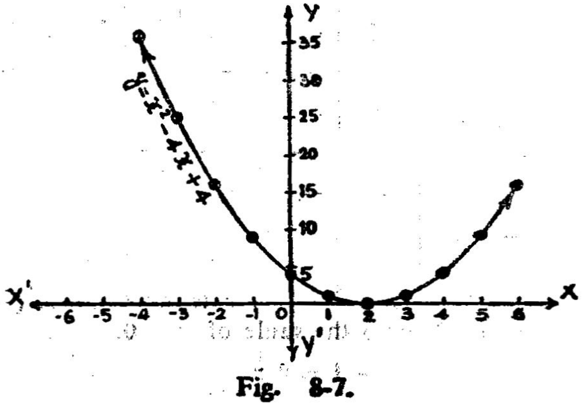


Fig. 8-7.

When $x = 2$, $y = 0$.

Hence $y = x^2 - 4x + 4 = 0 \Longrightarrow x = 2$

Example 3 :

Draw the graph of $y = x^2 - 2x - 3$.

Solution :

x	-4	-3	-2	-1	0	1	2	3	4	5	6
x^2	16	9	4	1	0	1	4	9	16	25	36
$-2x$	8	6	4	2	0	-2	-4	-6	-8	-10	-12
-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
$y = x^2 - 2x - 3$	21	12	5	0	-3	-4	-3	0	5	12	21

Scale :

On the X axis 1 cm $\rightarrow$ 1 unit.

On the Y axis 1 cm $\rightarrow$ 5 units.

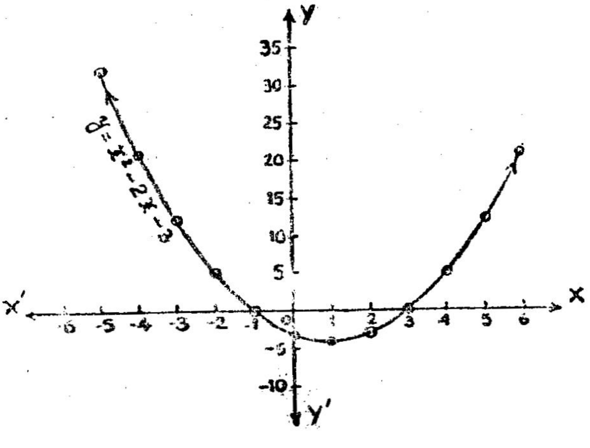


Fig. 8-8.

The graph of $y = x^2 - 2x - 3$ cuts the X axis at $(-1, 0)$, $(3, 0)$. For $x = -1$, or 3 the vaule of $y = 0$.

Hence, $y = x^2 - 2x - 3 = 0 \Longrightarrow x = -1$ or 3.

Exercise 8-1-4

Draw the graphs of the following trinomial quadratic functions.

(1) $y = x^2 + 2x + 7$ (2) $y = x^2 + 4x + 4$

(3) $y = x^2 - 2x - 15$ (4) $y = x^2 - 3x + 2$

8-1·5. Graph: Solution of Quadratic Equations

The solution set of $x^2 + ax + b = 0$ is the intersection of $y = x^2 + ax + b$ and $y = 0$. $y=0$ is the X axis. Hence, the solution set of $x^2+ax+b = 0$ can be found from the point where the graph cuts the X axis.

Graph No.	Fuction	The points where the graph cuts the X axis	Equation	Solution set in the set of real numbers
8·1	$y=x^2$	(0, 0)	$x^2=0$	{0}
8·2	$y=x^2+4$	Nil	$x^2+4=0$	{ }
8·3	$y=x^2-4$	(−2, 0) (2, 0)	$x^2-4=0$	{−2, 2}
8·4	$y=x^2+4x$	(−4, 0) (0, 0)	$x^2+4x=0$	{−4, 0}
8·5	$y=x^2-4x$	(0, 0), (4, 0)	$x^2-4x=0$	{0, 4}
8·6	$y=x^2+x+5$	Nil	$x^2+x+5=0$	{ }
8·7	$y=x^2-4x+4$	(2, 0)	$x^2-4x+4=0$	{2}
8·8	$y=x^2-2x+3$	(−1, 0) (3, 0)	$x^2-2x+3=0$	{−1, 3}

Note:

1. The x co-ordinates of the points where the graph of $y = x^2 + ax + b$ cuts the X axis will represent the solution set of $x^2 + ax + b = 0$.

2. The number of elements in the solution set of $x^2 + ax + b = 0$ cannot be more than 2.

Exercise 8-1-5

Find the solution set of the following equations by drawing graphs.

1. $x^2 - 9 = 0$ 2. $x^2 + 2x = 0$
3. $x^2 + x + 1 = 0$ 4. $x^2 - 2x - 8 = 0$

ANSWERS

(1) $\{3, -3\}$ (2) $\{0, -2\}$

(3) $\{\ \}$ (4) $\{4, -2\}$

8-2.1 Inequations

In the Ninth Standard, we have learnt to draw graphs of inequations. Let us recollect some of them.

Example 1 :

Draw the graph of the inequation $x + y \geqslant 6$; $x, y \in R$.

Construction :

To draw the graph of the inequation $x + y \geqslant 6$,

(1) Draw the graph of the straight line $x + y = 6$.

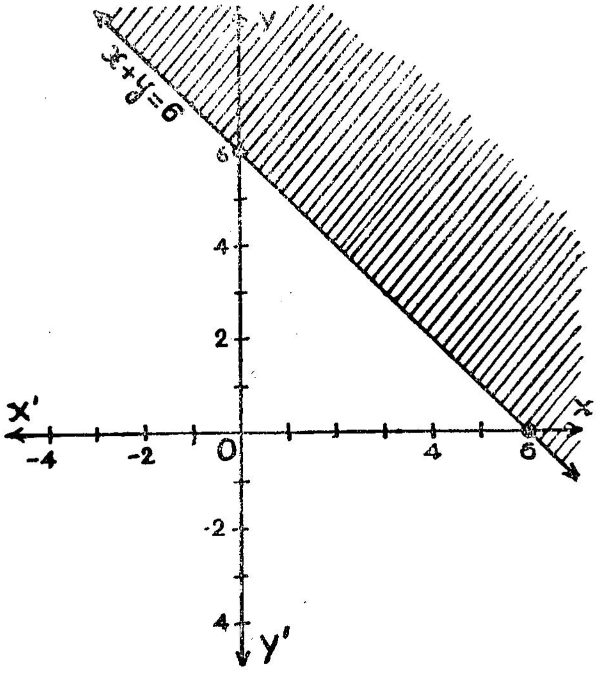


Fig. 8-9

(2) Take any point satisfying the relation $x + y > 6$

(e g.) (0, 7), (7,0) (4, 6) ... and find where the point lies. The plane inwhich the point lies is the graph of the inequation.

(3) Shade the region.

$x + y \geq 6$ is the union of $x + y = 6$ and $x + y > 6$. Hence the straight line showing $x + y = 6$ also lies in the shaded area.

Example 2 :

Draw the graph of the inequation $y - 2x \leq 4$ $\quad x, y \in R$

Construction :

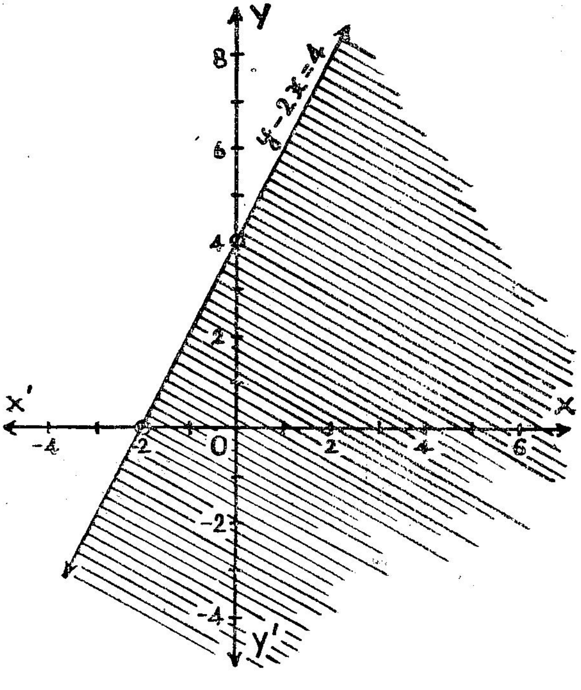


Fig 8-10

(1) Draw the graph of the straight line $y - 2x = 4$.

(2) Find the points which satisfy the relation $y - 2x < 4$

(e.g.) (0, 2), (1, 0) .. Find where these points lie. Their location will determine the plane of the graph.

(3) Shade the region. Note that $y - 2x = 4$ also lies in the shaded area.

Exercise 8-2.1

Draw the graphs of the following inequations. $(x, y \in R)$

(1) $x < 6$ (2) $y > 4$ (3) $2x + y < 5$ (4) $4x + y > 8$

8-2·2. Graph : Systems of Inequations

Example 1 :

Draw the graph of the inequations $x < 3$ and $y < 5$; $x, y \in R$.

Construction :

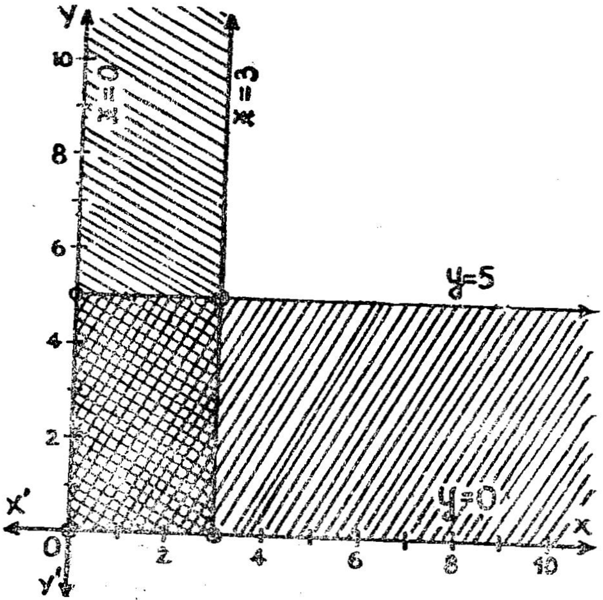


Fig. 8-11

(1) Draw the graph of the line $x = 3$.

(2) Determine the plane which refers to $x < 3$.

(3) Draw the graph of the line $y = 5$.

(4) Determine the plane which refers to $y < 5$.

(5) Shade the intersection of the planes relating to $x < 3$ and $y < 5$.

(6) The graphs of the system of inequations " $x < 3$ and $y < 5$ ", and the X and Y axes form a rectangle with vertices, (0, 0), (3, 0), (3, 5), (0, 5).

Example 2 :

Draw the graph of the inequations $x + y < 6$ and $2x + y < 8$; $x, y \in R$.

Construction :

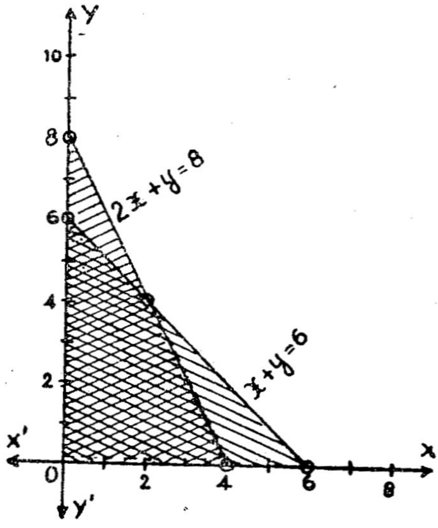


Fig. 8-12.

1. Draw the graph of the line $x + y = 6$.
2. Determine plane of the inequation $x + y < 6$.
3. Draw the graph of the line $2x + y = 8$.
4. Determine the plane of the inequation $2x + y < 8$.
5. Shade the intersection of the planes relating to "$x + y < 6$ and $2x + y < 8$".

6. The graphs of the inequations "$x + y < 6$ and $2x + y < 8$", the X axis and the Y axis form a quadrilateral. The coordinates of the vertices of the quadrilateral are respectively (0, 0), (4, 0), (2, 4), (0, 6).

Exercise 8-2-2

Draw the graph of the following system of inequations. $(x, y \in R)$

1. $x < 2$ and $y < 4$
2. $x + y < 6$ and $4x + y < 9$
3. $x + y \leq 5$ and $3x + 2y < 12$
4. $x + y < 10$ and $2x + 3y < 24$

8-2.3 Linear Programming : Maximizing

Let us now learn how the graphical representation of a set of inequalities helps us in our everyday life.

Example :

A farmer has enough space in his farm to rear 100 birds. He buys chickens at Rs. 2/- each and ducklings at Rs. 3/- each. He cannot spend more than Rs. 240/- for purchasing the birds. He rears them for some time and sells each chicken at Rs. 4/- and each duckling at Rs. 5/-. How many chickens and ducklings should be buy in order to get the maximum income ?

Solution :

	Chicken	Duckling	Constraint
Number	x	y	100
Price (Rs.)	2	3	240
Income (Rs.)	4	5	Maximum

Inequations : (1) $x + y < 100$

(2) $2x + 3y < 240$

(3) $x > 0$

(4) $y > 0$

[The four inequations got from the data are known as constraints.]

The income that the farmer gets is $P = 4x + 5y$. This is called the **Objective Function.** We have to find its maximum value.

In fig. 8-13, the four inequations form the inside of the quadrilateral *OABC*.

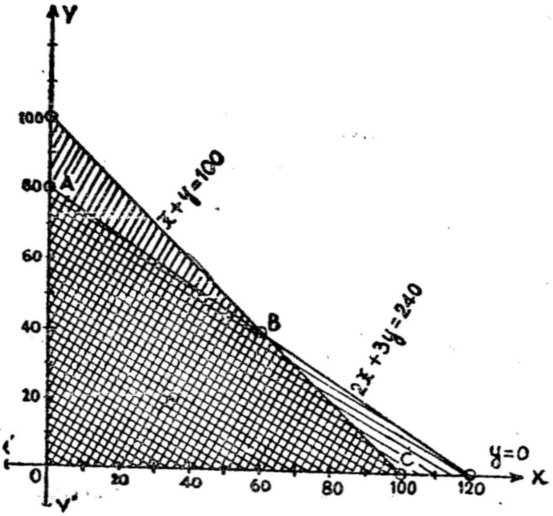


Fig. 8-13.

At O (0, 0) the income is Rs. (4.0 + 5.0) = Rs. 0

At A (0, 80) the income is Rs, (4.0 + 5.80) = Rs. 400

At B (60, 40) the income is Rs. (4.60 + 5.40) = Rs. 440

At C (100, 0) the income is Rs. (4.100 + 5.0) = Rs. 400

Hence the purchase of 60 chickens and 40 ducklings will give the maximum income of Rs. 440/-.

[Determining the maximum value by the above procedure is called **Maximizing.**]

Exercise 8.2-3

1. A cycle factory produces two types of cycles, one for the common use and the other for races. The factory cannot produce more than 200 cycles in all in a month. It takes 2 hours to make the ordinary cycles, and 3 hours for the special type. The factory cannot work for more than 480 hours in a month. The profit from the ordinary type is Rs. 20/- each and Rs. 30/- from each of the special type. To get the maximum profit, find the number of cycles to be produced in each category in a month.

2. Two kinds of cattlefeed are manufactured in a factory. The factory can produce 40 tonnes in all in a day. The expenses for the first kind is Rs. 3/- per ton and for the second Rs. 6/- per ton. The factory cannot afford to spend more than Rs. 180/- a day. The income from the first typs of cattle feed is Rs. 6 per ton and from the second Rs. 7 per ton. To get the maximum income, find the quantity to be produced in each type in a day.

3. Maximize $C = 3x + 2y$ subject to the following constraints $x > 0$, $y > 0$, $5x + y < 170$ and $x + y < 50$.

ANSWERS

1. 160 cycles in the special type, (or) 120 cycles in the common type and 80 in the special type Maximum profit Rs. 4800.

2. Cattle feed of the first type 20 tonnes, second type 20 tonnes. Maximum Income Rs. 260/-.

3. $x = 30$, $y = 20$, Maximum value of $C = 130$.

8-2·4. Linear Programming : Minimizing

Example :

A diet specialist prepares a new type of food by mixing two types of food containing a maximum of 16 units of Vitamin A and 14 units of Vitamin C. It is found that there are 2 units of Vitamin A per kg and 1 unit of Vitamin C per kg in the first type. In the second type, 1 unit of Vitamin A and 2 units of Vitamin C are found. The cost of the first type is Rs. 3 per kg and the second type is Rs. 4/- per kg. Find

the quantity of food to be purchased in each type so that on mixing, the price of the new type will be the minimum.

Solution :

	First type x kg.	Second Type y kg.	Constraint
Vitamin A	2	1	16
Vitamin B	1	2	14
Price (Rs.)	3	4	Minimum

Inequations : (1) $2x + y < 16$

(2) $x + 2y < 14$

(3) $x > 0$

(4) $y > 0$

Subject to the given constraints the cost of the type of food is given by $C = 3x + 4y$. We have to calculate the minimum value of this objective function.

In fig. 8-14, the four inequations form the quadsilateral $OABC$.

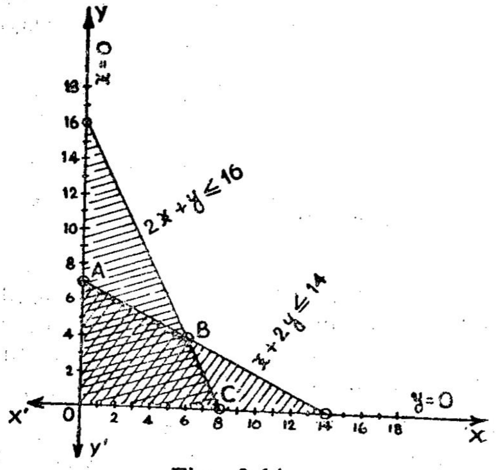


Fig. 8-14.

At 0 (0, 0) the price is (3.0 + 4.0) = Rs. 0

At *A* (0, 7) the price is (3.0 + 4.7) = Rs. 28

At *B* (6, 4) the price is (3.6 + 4.4) = Rs. 34

At *C* (8, 0) the price is (3.8 + 4.0) = Rs. 24

Hence the price of the new type of food to be the minimum, Rs. 24/-, it is enough if 8 kg of the first type of food is purchased.

(Determining the minimum value by the above procedure is called Minimizing.)

Exercise 8-2-4

1. Tailor *A* can make a maximum of 4 shirts and 4 trousers in a day. Tailor *B* can make a maximum of 6 shirts and 4 trousers in a day. The daily wages of *A* and *B* are respectively Rs. 15 and Rs. 20. The manager of a shop desires to make a maximum of 24 shirts and 20 trousers by paying minimum wages. Find the number of days he has to employ *A* and *B* for making the dresses.

2. A merchant does not like to spend more than Rs. 100 as cartage. The cartage for one bag of rice is Rs. 5 and for one bag of dhal is Rs. 4. He cannot store more than 22 bags at a time. The storage expense for one bag of rice is Rs. 2 and for one bag of dhal Re. 1. How many bags can be purchased in each variety to minimize the expense?

3. Minimize $C = 6x + 5y$ subject to the following constraints. $x + y < 10$, $y + 4x < 28$, $x > 0$ and $y > 0$.

ANSWERS

1. *A* for 5 days. Minimum wages Rs. 75.
2. 22 dhal bags. Minimum expense Rs. 22.
3. For $x = 7$, $y = 0$, the minimum value of C is 42.

REVIEW

1. To draw the graph of $y = ax^2 + bx + c$, give values to x and find the corresponding values of y. Then the ordered pairs of (x, y) are marked on the Cartesian plane. They

are joined by a smooth curve. This curve is the graph of the given function.

2. The x coordinates of points where the graph $y = x^2+ax+b$ cuts the X axis will represent the solution set of the equation $x^2 + ax + b = 0$. If the graph cuts the X axis at two points, then the solution set will have two elements. If it touches at one point, then the solution set will have one element. If it does not cut the X axis, then it will have no solution in R.

3. The straight line $y = mx + c$ divides the plane into two parts. One part represents the graph of the inequation $y > mx + c$ and the other $y < mx + c$. The graph of $y \geqslant mx + c$ is the union of the straight line $y = mx + c$ and the plane showing $y > mx + c$. The graph of $y \leqslant mx + c$ is the union of the straight line $y = mx + c$ and the plane representing $y < mx + c$.

4. The systems of inequations related by the word "and" represent the intersection of the two planes of the inequations.

5. Linear Programming helps to find the maximum or minimum value of a function with respect to a given set of constraints. This function which is maximized or minimized is called the **objective function.**

6. To solve graphically problems on Linear Programming

   (1) Draw the graphs of the constraints.

   (2) Mark the region represented by the solution set of the system of inequations.

   (3) Find the co-ordinates of the vertices of the polygon formed.

   (4) Substitute the values of those coordinates in the objective function. Find the particular value for which the value of the function is maximum or minimum.

SELF ASSESSMENT TEST

1. Solve graphically the quadratic equation $x^2 + 2x - 15 = 0$, $x \in R$ and find the solution set.

2. Maximize $C = 5x + 4y$ subject to the constraints $x \geq 0$, $y \geq 0$, $x + y \leq 30$ and $2x + y \leq 40$, $x, y \in R$.

3. Minimize $C = 7x + 4y$ subject to the constraints $x \geq 0$, $y \geq 0$, $x + y \leq 40$, and $2x + y \leq 60$, $x, y \in R$.

ANSWERS

1. $\{-5, 3\}$ 2. 130 ; $x = 10$, $y = 20$

3. 160, $x = 0$, $y = 40$.

NOT FOR EXAMINATION

MATHEMATICS CLUB — ACTIVITY 7

The Divine Descent

Divide a square of 16 cm as shown in fig. 8·15.

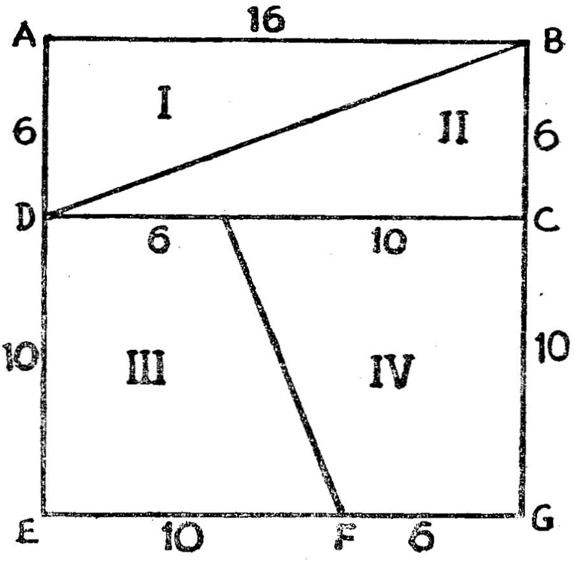


Fig. 8·15.

Area of the square = 16 × 16 = 256 cm^2. Fix the four pieces as shown in fig. 8-16.

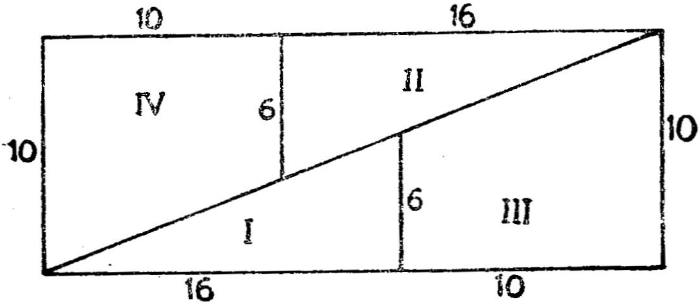


Fig. 8-16.

The area of this rectangle = 26 × 10 = 260 cm^2.

How did you get the extra 4 cm^2? Did it descend from heaven? Find out.

9. TRIGONOMETRY

Trigonometry deals with the relationship between the sides and angles of a triangle. It is closely related to the theorem of Pythagoras (*c* 540 BC). This branch of mathematics is used to solve problems in Surveying, Geography, Astronomy, Navigation, etc.,

Hipparchus (*c* 140 BC), the famous Greek astronomer is said to be the father of Trigonometry. Aryabhatta, Brahma gupta and Bhaskara of our country have also contributed much to the development of this branch of mathematics.

9.1.1 Sine Ratio

In fig. 9-1, $\angle CAB$ is an acute angle. Let its measure be θ.

If $\overrightarrow{BC} \perp \overrightarrow{AC}$, then $\triangle ABC$ is a right triangle.

And the ratio $\frac{BC}{AB} = \frac{a}{c}$ is said to be the **Sine ratio** of $\angle A$.

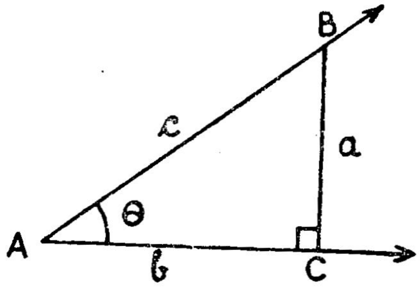


Fig. 9-1.

This is written as $\sin A = \frac{a}{b}$ or $\sin \theta = \frac{a}{c}$.

Now, let P be any point on $\overline{AB}$ and $\overline{PQ}$ be drawn perpendicular to $\overline{AC}$. $\triangle APQ$ is a right triangle. Also, $\triangle APQ ||| \triangle ABC$.

So,

$$\frac{PQ}{AP} = \frac{BC}{AB} = \sin\theta.$$

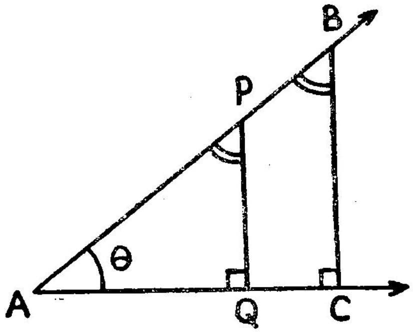


Fig. 9-2.

From this we know that **the sine ratio of an angle does not depend on the sides of the right triangle.**

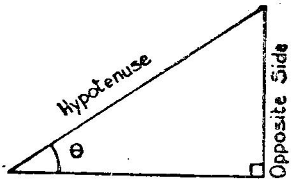


Fig. 9-3.

In general,

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

Example 1 :

In $\triangle ABC$, $m\angle C = 90°$, $BC = 3$, $AC = 4$, $AB =$

Find the sine ratio of $\angle A$.

Solution :

In ΔABC the side opposite to $\angle A = 3$

Hypotenuse $= 5$

$$\sin A = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{3}{5}$$

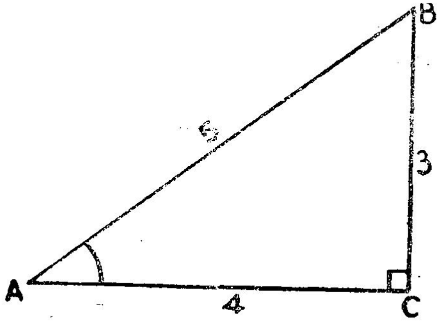


Fig 9-4.

Example 2 :

In a right triangle the sides containing the right angle are 5, 12 units. Find the sine ratio of the angle opposite to the side of 12 units.

Solution :

The length of the hypotenuse of the right triangle

$= \sqrt{5^2 + 12^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$ units.

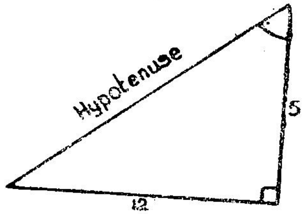


Fig. 9-5.

$\therefore$ Sine ratio of the angle opposite to the side of 12 units

$$= \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{12}{13}$$

Exercise 9·1·1

1. Find the value of sin θ in the following right triangles.

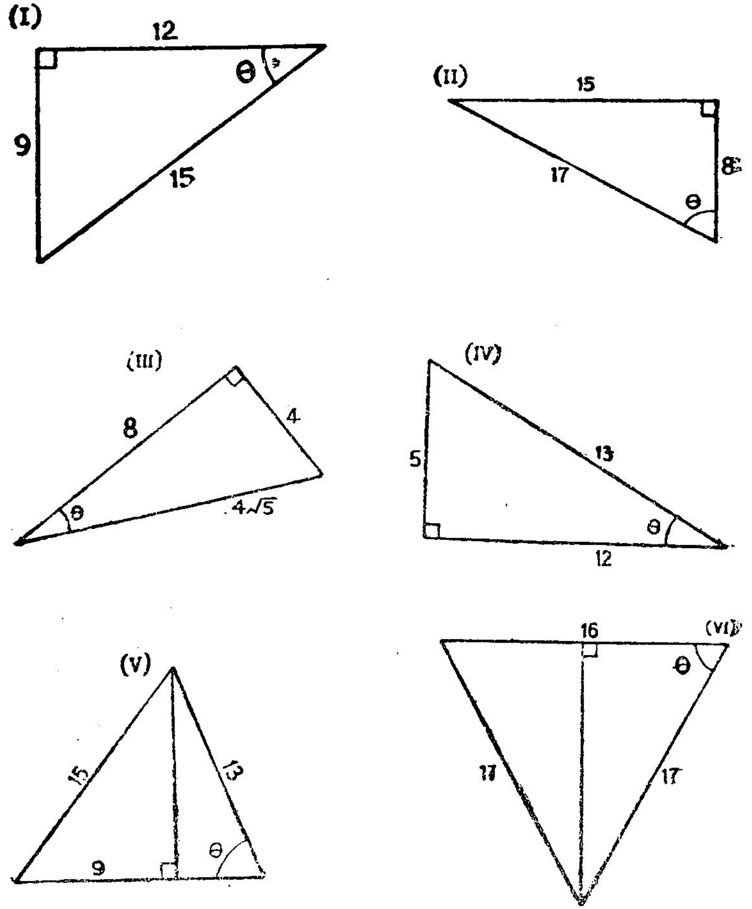


Fig. 9-6.

2. *A* person says that $\sin A = \dfrac{7}{3}$. Verify the truth of the statement. What do you infer about the maximum value of sin *A* ?

ANSWERS

(i) $\frac{3}{5}$ (ii) $\frac{15}{17}$. (iii) $\frac{1}{\sqrt{5}}$ (iv) $\frac{5}{13}$ (v) $\frac{12}{13}$ (vi) $\frac{15}{17}$

9-1-2. Cosine Ratio

In fig. 9-7, $\frac{AC}{AB} = \frac{b}{c}$ is the **Cosine ratio of** $\angle A$. This is written as $\cos A = \frac{b}{c}$ or $\cos \theta = \frac{b}{c}$.

In fig. 9-2, $\frac{AQ}{AP} = \frac{AC}{AB} = \cos \theta$.

From this, we learn that **the cosine ratio of an angle does not depend on the lengths of the sides of a right triangle.**

In general,

$$\cos \theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

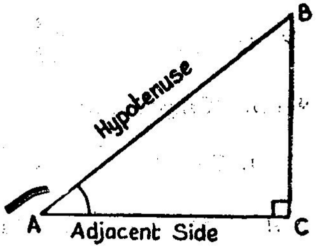


Fig. 9-7.

Example 1 :

In fig. 9-4, find the cosine ratio of $\angle A$.

Solution :

In $\triangle ABC$, the side opposite to $\angle A$ is 4 units, hypotenuse is 5 units.

$$\therefore \quad \cos A = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{4}{5}$$

Example 2 :

In a right triangle, one angle is θ. The side opposite to θ is 12. Hypotenuse is 13. Find $\cos \theta$.

Solution :

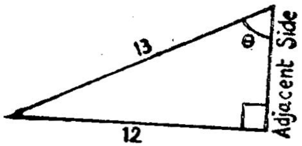


Fig. 9-8.

The adjacent side to θ

$= \sqrt{13^2 - 12^2}$

$= \sqrt{169 - 144}$

$= \sqrt{25}$

$= 5$

$$\cos \theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{5}{13}$$

Exercise 9-1.2

Find cos θ in the right triangles in Fig. 9-6.

ANSWERS

(i) $\frac{4}{5}$ (ii) $\frac{8}{17}$ (iii) $\frac{2}{\sqrt{5}}$ (iv) $\frac{12}{13}$

(v) $\frac{5}{13}$ (vi) $\frac{8}{17}$

9-1-3. Tangent ratio

In Fig. 9-9, $\frac{BC}{AC} = \frac{a}{b}$ is said to be the **tangent ratio** of $\angle A$. This is written as $\tan A = \frac{a}{b}$ or $\tan \theta = \frac{a}{b}$

In general,

$$\tan \theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

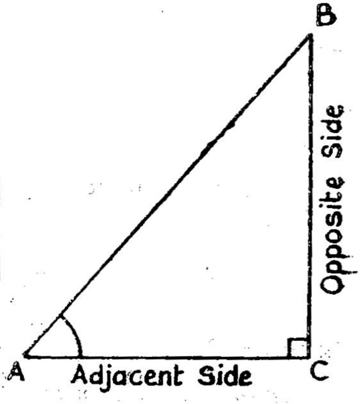


Fig. 9-9.

Example 1 :

In Fig. 9-4, find the tangent ratio of $\angle A$.

Solution :

In $\triangle\ ABC$, the side opposite to $\angle A$ is 3. Adjacent side is 4.

$$\therefore \quad \tan A = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{3}{4}$$

Example 2 :

In a right triangle one angle is θ. Its adjacent side is 5. Hypotenuse is 13. Find tan θ.

Solution :

The side opposite to θ

$= \sqrt{13^2 - 5^2}$

$= \sqrt{169 - 25}$

$= \sqrt{144}$

$= 12$

$\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$

$= \dfrac{12}{5} = 2{\cdot}4$

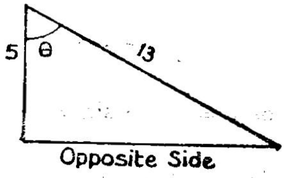


Fig. 9-10

Exercise 9-1-3

In Fig. 9-6, find tan θ in each of the right triangles.

ANSWERS

(i) $\frac{3}{4}$ (ii) $\frac{15}{8}$ (iii) $\frac{1}{2}$ (iv) $\frac{5}{12}$ (v) $\frac{12}{5}$

(vi) $\frac{15}{8}$

9-2.1. Complementary Angles : sine, cosine ratios

In Fig. 9-11, in the right triangle,

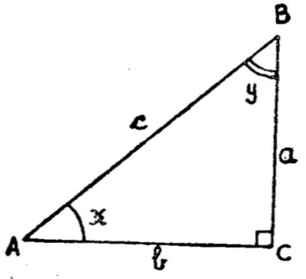


Fig. 9-11.

since $m\angle C = 90°$

$m\angle A + m\angle B = 90°$.

$\therefore$ $\angle A$ and $\angle B$ are **complementary angles.**

If $m\angle A = x$ and $m\angle B = y$,

then $x + y = 90°$

$x = 90° - y$ and

$y = 90° - x$

Now,

$$\sin x = \frac{a}{c}; \ \sin y = \frac{b}{c}$$

$$\cos x = \frac{b}{c}; \ \cos y = \frac{a}{c}$$

$$\sin x = \frac{a}{c} = \cos y$$

$$\cos x = \frac{b}{c} = \sin y$$

i. e. $\sin x = \cos y = \cos (90° - x)$

$\cos x = \sin y = \sin (90° - x)$

or $\sin x = \cos (90° - x)$

$\cos x = \sin (90° - x)$

In general,

$$\sin (90° - \theta) = \cos \theta$$
$$\cos (90° - \theta) = \sin \theta$$

The term cosine is the **shortened form** for 'sine of the complement' Hence **the cosine ratio of an angle is the sine ratio of its complement.**

Example 1 :

If $\sin 30° = 0.5$, find $\cos 60°$.

Solution :

$$\cos 60° = \sin (90° - 60°)$$
$$= \sin 30°$$
$$= 0{\cdot}5.$$

Example 2 :

Find the value of $\frac{\sin 20°}{\cos 70°}$

Solution :

$$\cos 70° = \sin (90° - 70°)$$
$$= \sin 20°$$
$$\therefore \quad \frac{\sin 20°}{\cos 70°} = \frac{\sin 20°}{\sin 20} = 1$$

Exercise 9-2.1.

1. If $\sin 60° = \frac{\sqrt{3}}{2}$, find $\cos 30°$.
2. Given $\cos 45° = \frac{1}{\sqrt{2}}$, find $\sin 45°$.
3. If $\sin 50° = \cos \theta$, find θ.
4. If $\cos 40° = \sin x$, find x.
5. Find the value of $\frac{\cos 80°}{\sin 10°}$
6. Show that $\frac{\sin 75°}{\cos 15°} = \frac{\cos 15°}{\sin 75°}$

ANSWERS

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{\sqrt{2}}$ (3) 40° (4) 50° (5) 1

9-2-2. Relation between sine and cosine ratios

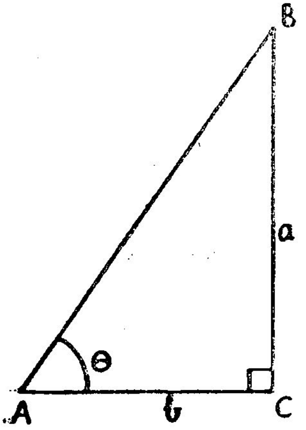


Fig. 9-12.

In a right triangle ABC, if $m\angle C = 90°$, then by Pyhagoras theorem,

$$a^2 + b^2 = c^2$$

$$\Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$\Rightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Usually $(\sin\theta)^2$, $(\cos\theta)^2$ are written as $\sin^2\theta$ and $\cos^2\theta$

Hence,

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Since $\sin^2\theta + \cos^2\theta = 1$,

$\sin^2\theta = 1 - \cos^2\theta$.

$\therefore\ \sin\theta = \sqrt{1 - \cos^2\theta}$

Also $\cos^2\theta = 1 - \sin^2\theta$

$\therefore\ \cos\theta = \sqrt{1 - \sin^2\theta}$

Example 1 :

In a right triangle, $\sin\theta = \frac{3}{5}$. Find $\cos\theta$.

Solution :

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

i. e. $\cos\theta = \sqrt{1 - \sin^2\theta}$

$= \sqrt{1 - \left(\frac{3}{5}\right)^2}$

$= \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25 - 9}{25}}$

$= \sqrt{\frac{16}{25}} = \frac{4}{5}$

Example 2 :

Simplify : $\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta$

Solution :

$\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta$

$= (\cos^2\theta)^2 + (\sin^2\theta)^2 + 2\sin^2\theta\cos^2\theta$

$= (\cos^2\theta + \sin^2\theta)^2$

$= 1^2$

$= 1$

Exercise 9-2-2

1. In a right triangle $\cos\theta = \frac{5}{13}$. Find $\sin\theta$.

2. If $x = 4\cos\theta$ and $y = 5\sin\theta$, find the value of $\frac{x^2}{16} + \frac{y^2}{25}$

3. Simplify : $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$.

4. Simplify : $\sin\theta\cos(90° - \theta) + \cos\theta\sin(90° - \theta)$.

5. Find the value of $\cos\theta\cos(90° - \theta) - \sin\theta\sin(90° - \theta)$.

6. Find the value of : $(1 + \sin\theta)(1 - \sin\theta) - \cos^2\theta$.

ANSWERS

(1) $\frac{12}{13}$ (2) 1 (3) 2 (4) 1 (5) 0 (6) 0

9-2.3. Relation between sine, cosine and tangent ratios

In Fig. 9-12, in $\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$, $\tan\theta = \frac{a}{b}$

Now, $\tan\theta = \frac{a}{b} = \frac{\left(\frac{a}{c}\right)}{\left(\frac{b}{c}\right)} = \frac{\sin\theta}{\cos\theta}$

Also if $\tan\theta = \frac{a}{b}$,

$\sin\theta = \frac{a}{\sqrt{a^2 + b^2}}$ and $\cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$

Example 1:

In a right triangle, if $\tan\theta = \sqrt{3}$, find $\sin\theta$.

Solution:

$$\tan\theta = \frac{a}{b} = \sqrt{3}$$

$$a = \sqrt{3},\ b = 1$$

$$c^2 = a^2 + b^2 = (\sqrt{3})^2 + 1^2$$

$$= 3 + 1 = 4$$

$$c = \sqrt{4} = 2$$

$$\sin\theta = \frac{a}{c} = \frac{\sqrt{3}}{2}$$

Example 2:

In a right triangle $\tan\theta = \frac{8}{15}$ Find $\sin\theta$ and $\cos\theta$.

Solution:

In Fig 9-13, length of the hypotenuse.

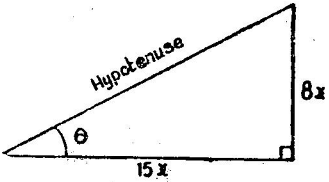


Fig. 9-13.

$= \sqrt{(8x)^2 + (15x)^2}$

$= \sqrt{64x^2 + 225x^2}$

$= \sqrt{289x^2}$

$= 17x.$

$$\therefore \sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{8x}{17x} = \frac{8}{17}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{15x}{17x} = \frac{15}{17}$$

Exercise 9-2.3

θ, is an angle in a right triangle. Complete the table using the given measurements.

No.	$\sin\theta$	$\cos\theta$	$\tan\theta$
i	$\frac{3}{5}$		
ii		0.6	
iii			$2\frac{2}{5}$
iv		$\frac{15}{17}$	
v	$\frac{a}{b}$		
vi			$\frac{p}{q}$

ANSWERS

(i) $\frac{4}{5}$, $\frac{3}{4}$ (ii) 0·8, 1·33

(iii) $\frac{12}{13}$, $\frac{5}{13}$ (iv) $\frac{8}{17}$, $\frac{8}{15}$

(v) $\frac{\sqrt{b^2 - a^2}}{b}$, $\frac{a}{\sqrt{b^2 - a^2}}$

(iv) $\frac{p}{\sqrt{p^2 + q^2}}$, $\frac{q}{\sqrt{p^2 + q^2}}$

9-3.1. Trigonometric ratios of angles of measures 30°, 45°, and 60°

(i) $\Delta\ ABC$ is an isosceles right triangle. $m\angle C = 90°$, $m\angle A = m\angle B = 45°$. If $a = b = 1$, $c = \sqrt{2}$

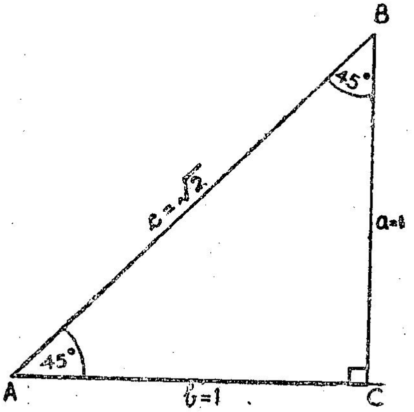


Fig. 9-14.

Therefore,

$$\text{Sin } 45° = \sin A$$

$$= \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1{\cdot}414}{2}$$

$$= 0.707$$

$\cos 45° = \cos A$

$$= \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1{\cdot}414}{2}$$

$$= 0{\cdot}707$$

$$\tan 45° = \tan A = \frac{a}{b} = \frac{1}{1} = 1$$

(ii) In a right triangle PQR, $m\angle R = 90°$

Let $m\angle P = 60°$ and $m\angle Q = 30°$

You have studied in the Ninth Standard that if the angles of a triangle are 30°, 60° and 90°, the sides opposite to them are respectively a, $a\sqrt{3}$ and $2a$.

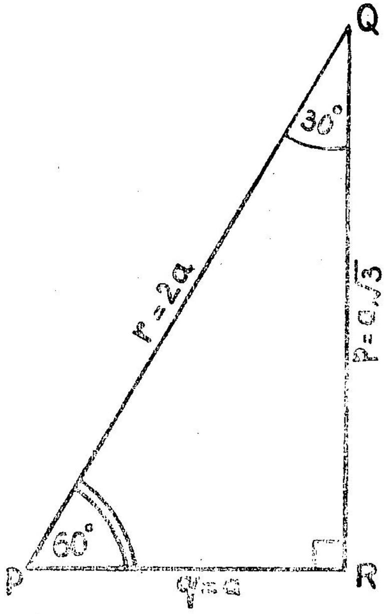


Fig. 9-15

Hence, $p = a\sqrt{3}$, $q = a$, $r = 2a$

$$\sin 30° = \cos Q = \frac{q}{r} = \frac{a}{2a} = \frac{1}{2} = 0{\cdot}5$$

$$\cos 30° = \cos Q = \frac{p}{r} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$= \frac{1\ 732}{2} = 0{\cdot}866$$

$$\tan 30° = \tan Q = \frac{q}{p} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$= \frac{1{\cdot}732}{3} = 0{\cdot}577$$

(iii) $$\sin 60° = \sin P = \frac{p}{r} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} = \frac{1{\cdot}732}{2}$$

$$= 0{\cdot}866$$

$$\cos 60° = \cos P = \frac{q}{r} = \frac{1a}{2a} = \frac{1}{2} = 0{\cdot}5$$

$$\tan 60° = \tan P = \frac{p}{q} = \frac{a\sqrt{3}}{a} = \sqrt{3} = 1{\cdot}732$$

These results are tabulated as follows.

θ	sin θ	cos θ	tan θ
30°	0·500	0·866	0·577
45°	0·707	0·707	1 000
60°	0·866	0·500	1·732

We can infer from the table that the values of sin θ and tan θ increase as θ increases and the value of cos θ decreases as θ increases.

Further, we learn the following :

$$\sin 30° = \frac{1}{2} = \cos 60° = \cos (90° - 30°)$$

$$\sin 45° = \frac{1}{\sqrt{2}} = \cos 45° = \cos (90° - 45°)$$

$$\sin 60° = \frac{\sqrt{3}}{2} = \cos 30° = \cos (90° - 60°)$$

Exercise 9-3-1

1. Complete the following table :

θ	30°	45°	60°
sin θ		$\frac{1}{\sqrt{2}}$	
cos θ	$\frac{\sqrt{3}}{2}$		
tan θ			$\sqrt{3}$

2. Verify whether the values noted in the following table are correct :

θ	306	45°	60°
sin θ	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$
cos θ	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$
tan θ	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{2}}$	$\sqrt{\frac{3}{1}}$

9.3.2. Trigonometric ratios : Problems

Problems similar to the following questions can be easily solved by substituting the values of trigonometric ratios of angles.

Example 1 :

Find the value of 2 sin 45° cos 45° tan 45°

Solution :

$$\sin 45° = \frac{1}{\sqrt{2}} = \cos 45°$$

$$\tan 45° = 1$$

$$2 \sin 45° \cos 45° \tan 45° = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times 1$$

$$= 2 \times \frac{1}{2} = 1.$$

Example 2 :

Prove : $\sin 30° \cos 60° + \cos 30° \sin 60° = 1$

Solution :

$$\sin 30° = \frac{1}{2} = \cos 60°$$

$$\sin 60° = \frac{\sqrt{3}}{2} = \cos 30°$$

By substituting these values we get

$$\sin 30° \cos 60° + \cos 30° \sin 60° = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4} + \frac{3}{4} = 1$$

Exercise 9-3-2

Find the value of the following :

1. $\sin 45° + \cos 45° + \tan 45°$
2. $\sin 45° \tan 45° + \cos^2 45° \tan 45°$
3. $\sin^2 45° + \cos^2 45°$
4. $\sin 30° \cos^2 30° \tan 30°$
5. $1 - \sin^2 30°$
6. $1 + \tan^2 30°$
7. $\dfrac{\tan 60°}{\sin 60° + \cos 60°}$

8. $\dfrac{\sin 60° - \cos 60°}{\tan 60°}$

9. $\tan 30° \tan 60°$

10. $\tan^2 45° + \tan^2 60°$

11. $\sin^2 30° + \cos^2 45° + \tan^2 60°$

12. $\tan^2 30° + \tan^2 45° + \tan^2 60°$

ANSWERS

(1) $1 + \sqrt{2}$ (2) $\sqrt{2}$ (3) 1 (4) $\frac{1}{4}$ (5) $\frac{3}{4}$

(6) $1\frac{1}{3}$ (7) $\dfrac{2\sqrt{3}}{1 + \sqrt{3}}$ (8) $\dfrac{\sqrt{3} - 1}{2\sqrt{3}}$ (9) 1

(10) $1\frac{3}{4}$ (11) $3\frac{3}{4}$ (12) $4\frac{1}{3}$

9-4. Trigonometric ratios : table

We have found out the values of trigonometric ratios of angles whose measures are 30°, 45°, and 60° The trigono 90° metric ratios of any angle whose measure is between 0° and 90° have been calculated and given in the form of a table. The sine, cosine and tangent ratios of angles whose measures are whole numbers are given correct to the fourth decimal place. Using the table given in the appendix to this book, we can solve problems.

Example 1 :

Using the table find the value of sin 35°. Find also the angle which has the same value for its cosine ratio.

Solution :

(i) Look at the table and read 35 down the column showing angles A. Move your finger towards right and below sin A. You find the value ·5736.

(ii) Move your finger down the **column** cos A and see the figure ·5736. Now move your finger towards left and read the entry under angle A. You get 55°.

Therefore, $\cdot 5736 = \cos 55°$

Example 2 :

Find the value of

$$\sin 13° - \cos 31° + \tan 44°$$

Solution :

$\sin 13° = \cdot 2250$

$\cos 31° = \cdot 8572$

$\tan 44° = \cdot 9657$

$\therefore \quad \sin 13° - \cos 31° + \tan 44°$

$= 0\cdot 2250 - 0\cdot 8572 + 0\cdot 9657$

$= 1\cdot 1907 - 0\cdot 8572$

$= 0\cdot 3335$

Exercise 9-4

1. Using the trigonometric tables, **find the value of**

(i) cos 65°	(ii) tan 40°	(iii) cos 7°
(iv) sin 73°	(v) tan 66°	(vi) sin 37°

2. Find the value of A if

(i) $\sin A = 0\cdot 1736$

(ii) $\tan A = 11\cdot 43$

(iii) $\cos A = 0\cdot 9336$

(iv) $\sin A = 0\cdot 8480$

(v) $\tan A = 0\cdot 7002$

(vi) $\cos A = 0\cdot 3256$

3. Find the value of :

(i) $\sin 40^\circ - \cos 50^\circ + \tan 81^\circ$

(ii) $\tan 42^\circ - \tan 24^\circ + \tan 88^\circ$

ANSWERS

1. (i) ·4226 (ii) ·8391 (iii) ·9925
   (iv) ·9563 (v) 2·2460 (vi) ·6018
2. (i) 10° (ii) 85° (iii) 21°
   (iv) 58° (v) 35° (vi) 71°
3. (i) 6·3138 (ii) 29·0952

9-5-1. Heights and Distances

Using trigonometric ratios, we can solve problems on heights and distances. For example we can find the height of a building or a hill or we can find the width of a river, and so on.

Example 1 :

A pendulum of length 30 cm deviates to a maximum of 17° from the vertical. What will be the distance of the bob of the pendulum from the point of suspension when it is at one extreme?

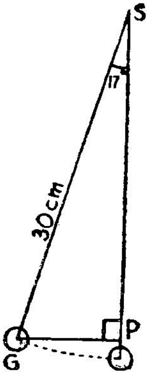


Fig. 9-16.

Solution :

As shown in fig. 9-16, the bob of the pendulum is at a distance of SP from the point of suspension.

$$\text{Distance} = SP$$

$$\frac{SP}{SG} = \cos S$$

$$\begin{aligned} SP &= SG \cos S \\ &= 33 \cos 17^\circ \\ &= 30 \times 0{\cdot}9563 \\ &= 28{\cdot}689 \\ &= 28{\cdot}7 \text{ cm} \end{aligned}$$

(correct to one decimal place)

Example 2 :

A man tries to swim across a river of 40 m width. But in reality he swims along the current at an angle of 65° to the vertical. What will be the distance between the place of landing and the point directly opposite to the starting point ?

Solution :

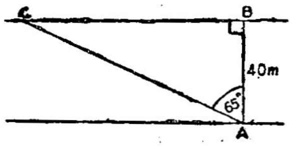


Fig. 9-17

Let the distance between the place of landing and the point directly opposite to the starting point be *BC.* (as shown in Fig. 9·17)

$$\frac{BC}{AB} = \tan 65^\circ$$

$$BC = AB \tan 65^\circ$$

$$= 40 \tan 65^\circ$$

$$= 40 \times 2{\cdot}1445$$

$$= 85{\cdot}78$$

$$= 86 \text{ m (correct to a metre)}$$

Exercise 9-5-1

1. Find the angle subtended at the centre by the chord of length 8 cm in a circle of diameter 10 cm.

2. The diagonal of a rectangle makes an angle of 50° with one side. If the length of the diagonal is 4 cm, find the length and breadth of the rectangle.

3. To find the distance between two points on opposite sides of a tank, a person takes measurements as shown in the adjoining figure. Find the distance.

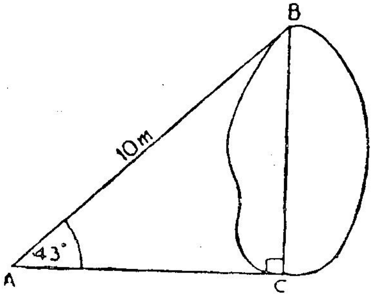


Fig. 9-18

4. The base of an isosceles triangle is 12 cm. Each of its base angles is 71°. Find its height.

5. The diagonals of a rhombus are 8 cm and 6 cm respectively. Find its angles.

6. In $\triangle ABC$, height $AD = 4$ cm, $m\angle B = 50°$, $m\angle C = 30°$ Find BC.

ANSWERS

(1) 54° (2) 3·1 cm, 2·6 cm (3) 6·82 m (4) 17·4 cm

(5) 74°, 106°, 74°, 106° (6) 10·3 cm.

9-5·2. Angle of Elevation I

Fig. 9·19 shows a man observing the top of a tree. The angle between the horizontal at the point of observation and the line of vision (that is the line connecting the eye to the point observed) is called the **angle of elevation.**

Some practical applications of this idea are given in the following examples.

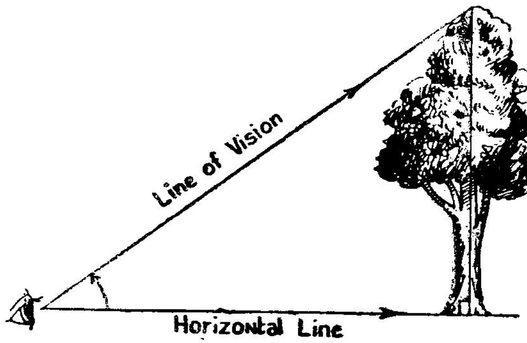


Fig. 9-19.

Example 1:

A railway track is laid on a mountain at an elevation of 10°. To what height has the track been raised in the course of a distance of 400 m?

Solution :

As shown in fig. 9-20. AB is the height to which the track has been raised.

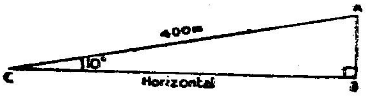


Fig. 9-20.

$$\frac{AB}{AC} = \sin 10°$$

$$\therefore \quad AB = AC \sin 10°$$

$$= 400 \times 0{\cdot}1736$$

$$= 69{\cdot}44 \text{ m}$$

Example 2 :

The rope supporting a flag post is fixed to the ground 8 m away from the post making an angle of elevation of 60° to the ground. Calculate the length of the rope.

Solution :

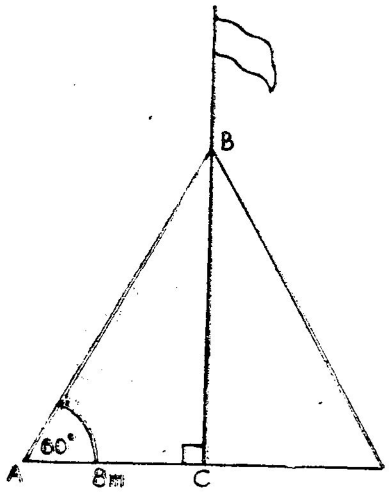


Fig. 9-21

Length of the rope in the figure is AB.

$$\cos 60° = \frac{AC}{AB}$$

$$AB = \frac{8}{\cos 60°} = \frac{8}{0 \cdot 5}$$

$$= 16 \text{ m}$$

Exercise 9-5-2

1. The string of a kite is 100 m long. It makes an angle of elevation of 50° with the horizontal. Find the height of the kite. (Assume that there is no sag in the string.)

2. A vertical pole, 6 m high is secured from its top by a taut rope fixed to the ground. The rope makes an angle of 30° with the ground. Calculate the distance a juggler has to walk on the rope in order to reach the top of the pole.

3. A scout climbs up the slope of a hill which is found to be at an inclination of 35° and travels a distance of 200 m. Calculate the corresponding horizontal distance covered by the scout.

4. A boat is fixed to the bank of a river with a rope 5 m long. The rope is taut and makes an angle of 40° with the horizontal. Calculate the horizontal distance between the boat and the bank.

5. A ladder leans against a vertical wall and makes an angle of 70° with the ground. The foot of the laddder is 3 m away from the wall. How high up the wall does the ladder reach?

6. Find the angle of elevation of the source of light if the shadow cast by a pole of 4 m high is 3 m long.

ANSWERS

(1) 76·6 m	(2) 12 m	(3) 163·84 m
(4) 3·83 m	(5) 8·24 m	(6) 48°

9-5.3. Angle of elevation II

Example 1 :

A person of height 1·5 m observes the measure of the angle of elevation of the top of a tower from a place 15 m from its foot as 53°. Find the height of the tower.

Solution :

In Fig. 9-22, the height of the tower is $(x + 1{\cdot}5)$ m.

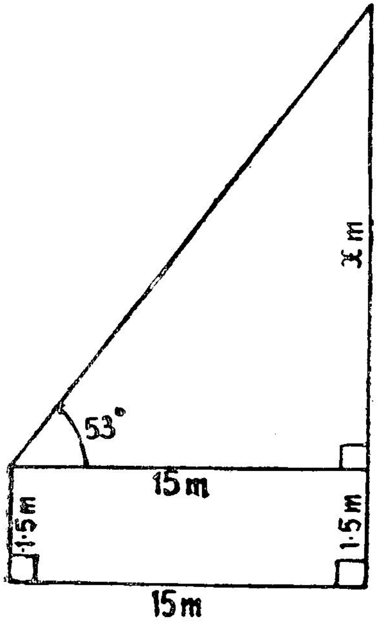


Fig. 9-22.

$$\tan 53° = \frac{x}{15}$$

$$\therefore \quad x = 15 \tan 53°$$

$$= 15 \times 1{\cdot}3270$$

$$= 19{\cdot}9050$$

$\therefore$ The height of the tower $= 19{\cdot}9050 + 1{\cdot}5$

$= 21{\cdot}4050$

$= 21{\cdot}4$ m

(correct to one decimal)

Example 2 :

A flagstaff is fixed on the top of a building. From a point on the ground 50 m away from the base of the building, the top and foot of the flagstaff are observed at an angle of elevation of 46° and 40° respectively. Find the height of the flagstaff.

Solution :

In the figure the height of the building is x m and that of the flagstaff is y m.

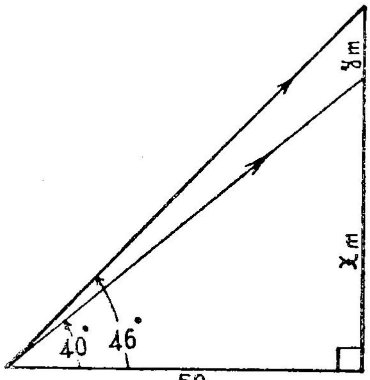


Fig. 9-23.

The height of the building, $x = 50 \tan 40°$

$= 50 \times 0{\cdot}8391$

$= 41{\cdot}9550$ m

The height of the building and the flagstaff is $(x + y)$ m.

$x + y = 50 \tan 46°$

$= 50 \times 1{\cdot}0355$

$= 51{\cdot}775$

Hence the height of the flagstaff $= 51{\cdot}775 - 41{\cdot}955$

$= 9{\cdot}82$ m

Exercies 9-5.3.

1. A person standing on the bank of a river observes the angle of elevation of the top of a tree on the opposite bank as 45°. When he moves 3 m backward, in the same line he finds the angle to be 30°. Find the width of the river.

2. The height of a lighthouse is 30 m. A person observes the angle of elevation of the top of the lighthouse from a place on the level ground as 60°. He moves away from the tower and halts at the place from where the angle of elevation is 30°. Calculate the distance between the two places from where the person observed the top of the lighthouse.

3. A ladder 20 m long is fixed to the base of a fire engine which is 2 m above the ground level. The ladder makes an angle of elevation of 70° with the base. How high can the top of the ladder reach from the ground?

4. The angles of elevation of a hill as observed from the bottom and top of a tower are 60° and 45°. The height of the hill is 100 m. Find the height of the tower.

5. There is a tower in between and in a line with two places situated at a distance of 200 m. The angles of elevation of the tower from these two places are 30° and 45°. Find the height of the tower.

6. The height of a lighthouse is 25 m. Two ships lie on opposite sides of the lighthouse. The angles of elevation of the tower as seen from these ships are 60° and 30°. Find the distance between the ships, if the lighthouse and the two ships are in a line.

ANSWERS

(1) 4·10 m	(2) 34·64 m	(3) 20·79 m
(4) 42·27 m	(5) 73·2 m	(6) 57·73 m

9-5.4. Angle of Depression

A man observing a lotus is shown in fig. 9-24. The angle between the horizontal and the line of vision is said to be the **angle of depression.**

Heights and distances can also be measured using this idea.

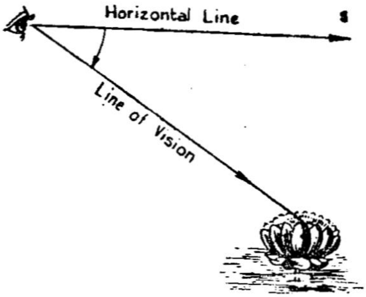


Fig. 9-24

Example 1 :

From the top of a rock 130 m high, the angle of depression of the top of a tower is 27°. If the distance between the tower and the rock is 40 m, calculate the height of the tower

Solution :

In Fig. 9-25 the height of the tower is x m.

$$\tan 27^\circ = \frac{130 - x}{40}$$

$$0{\cdot}5095 = \frac{130 - x}{40}$$

$$20{\cdot}38 = 130 - x$$
$$x = 130 - 20{\cdot}38$$
$$= 109\ 62$$

The height of the tower is 109·62 m

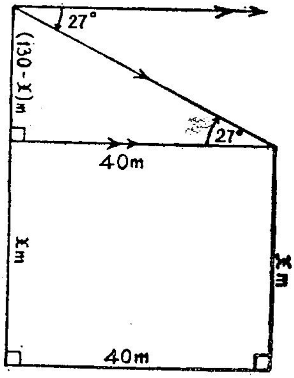


Fig. 9-25

Example 2 :

Two boats lie due east of a lighthouse. The angles of depression of the boats from the top of the lighthouse are respectively 45° and 43°. If the distance between the two boats is 2 m, find the height of the lighthouse.

Solution:

In the figure the height of the light house is x m.

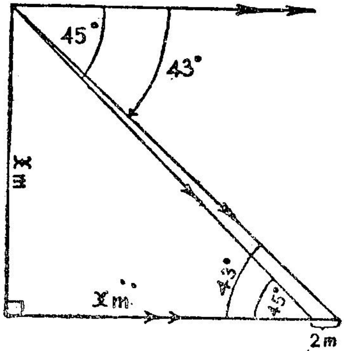


Fig 9-26

$$\tan 43° = \frac{x}{x+2}$$

$$0{\cdot}9325 = \frac{x}{x+2}$$

$$0{\cdot}9325\,x + 1{\cdot}8650 = x$$

$$x - 0{\cdot}9325\,x = 1{\cdot}8650$$

$$0{\cdot}0675\,x = 1{\cdot}8650$$

$$x = \frac{1{\cdot}8650}{0{\cdot}0675}$$

$$= 27{\cdot}63$$

The height of the lighthouse is 27·63 m

Exercise 9·5·4

1. The angle of depression of a point on level ground from the top of a building is 30°. If the point is 30 m away from the foot of the building, find the height of the building.

2. From the top of a lighthouse 30 m high, the angle of depression of a ship is observed to be 60°. Find the distance between the lighthouse and the ship.

3. From the top of a hill, the angles of depression of two consecutive kilometre stones on a horizontal road running due west from the base of the hill are 45° and 10°. Find the height of the hill.

4. A steep rock 150 m high above the sea level stands on the seashore. Two ships lie on opposite sides of the rock. The angles of depression of the ships from the top of the rock are 45° and 30°. Find the distance between the ships, if the line joining them passes through the foot of the rock.

5. The angles of depression of the top and bottom of a tree observed from the top of a building are 30° and 60° respectively. If the height of the building is 18 m, find the height of the tree.

6. An observer is an air craft flying at an altitude of 1500 m above one end of a bridge, finds the farther end of the bridge to be at an angle of depression of 86°. Find the length of the bridge.

ANSWERS

(1) 17·32 m (2) 17·32 m (3) 214·03 m (4) 409·80 m

(5) 12 m (6) 104·9 m

REVIEW

1. In $\triangle ABC$, if $\angle C$ is a right angle,

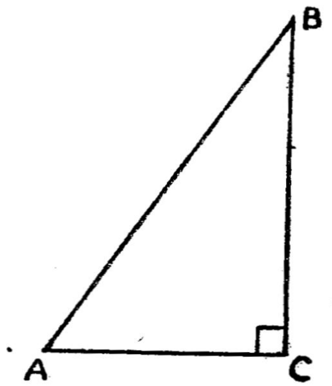


Fig. 9-27.

$$\sin A = \frac{a}{c} = \cos B\,; \quad \tan A = \frac{a}{b}$$

$$\sin B = \frac{b}{c} = \cos A\,; \quad \tan B = \frac{b}{a}$$

A and *B* are complementary angles.

$\cos B = \sin A$; $\cos A = \sin B$

Cosine is the shortened form of the 'sine of the complement.'

2. $\sin^2\theta + \cos^2\theta = 1$; $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

3. $\sin 30° = \dfrac{1}{2}$; $\sin 45° = \dfrac{1}{\sqrt{2}}$; $\sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos 30° = \dfrac{\sqrt{3}}{2}$; $\cos 45° = \dfrac{1}{\sqrt{2}}$; $\cos 60° = \dfrac{1}{2}$

$\tan 30° = \dfrac{1}{\sqrt{3}}$; $\tan 45° = 1$; $\tan 60° = \sqrt{3}$

4. The angle between the horizontal at the point of observation and the line of vision is called the horizontal (i) the angle of elevation when the point observed is above and (ii) the angle of depression when the point observed is below the horizontal.

The trigonometric ratios of the angles of elevation and depression can be used to solve problems on heights and distances.

SELF ASSESSMENT TEST

1. If $\cos\theta = 0{\cdot}8$, find $\sin\theta$ and $\tan\theta$.
2. If $\cos 55° = \sin\theta$, find θ.
3. Simplify :

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} - \frac{2}{\cos^2\theta}$$

4. Find the value of :

$$\sin^2 60° + \cos^2 45° + \tan^2 30°$$

5. Find the value of :

$$\sin 27° - \cos 63° + \tan 62°$$

6. A certain disabled plane can glide at an angle of 10° with the horizontal. If it starts gliding at an altitude of 7025 m, can it reach a landing field 20 km away ?

ANSWERS

(1) 0.6; 0.75	(2) 35°	(3) 0
(4) $\frac{19}{12}$	(5) 1.8807	(6) No

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
1·0	·0000	·0043	·0086	·0128	·0170	·0212	·0253	·0294	·0334	·0374	4	8	12	17	21	25	29	33	37
1·1	·0414	·0453	·0492	·0531	·0569	·0607	·0645	·0682	·0719	·0755	4	8	11	15	19	23	26	30	34
1·2	·0792	·0828	·0864	·0899	·0934	·0969	·1004	·1038	·1072	·1106	3	7	10	14	17	21	24	28	31
1·3	·1139	·1173	·1206	·1239	·1271	·1303	·1335	·1367	·1399	·1430	3	6	10	13	16	19	23	26	29
1·4	·1461	·1492	·1523	·1553	·1584	·1614	·1644	·1673	·1703	·1732	3	6	9	12	15	18	21	24	27
1·5	·1761	·1790	·1818	·1847	·1875	·1903	·1931	·1959	·1987	·2014	3	6	8	11	14	17	20	22	25
1·6	·2041	·2068	·2095	·2122	·2148	·2175	·2201	·2227	·2253	·2279	3	5	8	11	13	16	18	21	24
1·7	·2304	·2330	·2355	·2380	·2405	·2430	·2455	·2480	·2504	·2529	2	5	7	10	12	15	17	20	22
1·8	·2553	·2577	·2601	·2625	·2648	·2672	·2695	·2718	·2742	·2765	2	5	7	9	12	14	16	19	21
1·9	·2788	·2810	·2833	·2856	·2878	·2900	·2923	·2945	·2967	·2989	2	4	7	9	11	13	16	18	20
2·0	·3010	·3032	·3054	·3075	·3096	·3118	·3139	·3160	·3181	·3201	2	4	6	8	11	13	15	17	19
2·1	·3222	·3243	·3263	·3284	·3304	·3324	·3345	·3365	·3385	·3404	2	4	6	8	10	12	14	16	18
2·2	·3424	·3444	·3464	·3483	·3502	·3522	·3541	·3560	·3579	·3598	2	4	6	8	10	12	14	15	17
2·3	·3617	·3636	·3655	·3674	·3692	·3711	·3729	·3747	·3766	·3784	2	4	6	7	9	11	13	15	17
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2·5	·3979	·3997	·4014	·4031	·4048	·4065	·4082	·4099	·4116	·4133	2	3	5	7	9	10	12	14	15
2·6	·4150	·4166	·4183	·4200	·4216	·4232	·4249	·4265	·4281	·4298	2	3	5	7	8	10	11	13	15
2·7	·4314	·4330	·4346	·4362	·4378	·4393	·4409	·4425	·4440	·4456	2	3	5	6	8	9	11	13	14
2·8	·4472	·4487	·4502	·4518	·4533	·4548	·4564	·4579	·4594	·4609	2	3	5	6	8	9	11	12	14
2·9	·4624	·4639	·4654	·4669	·4683	·4698	·4713	·4728	·4742	·4757	1	3	4	6	7	9	10	12	13
3·0	·4771	·4786	·4800	·4814	·4829	·4843	·4857	·4871	·4886	·4900	1	3	4	6	7	9	10	11	13
3·1	·4914	·4928	·4942	·4955	·4969	·4983	·4997	·5011	·5024	·5038	1	3	4	6	7	8	10	11	12
3·2	·5051	·5065	·5079	·5092	·5105	·5119	·5132	·5145	·5159	·5172	1	3	4	5	7	8	9	11	12
3·3	·5185	·5198	·5211	·5224	·5237	·5250	·5263	·5276	·5289	·5302	1	3	4	5	6	8	9	10	12
3·4	·5315	·5328	·5340	·5353	·5366	·5378	·5391	·5403	·5416	·5428	1	3	4	5	6	8	9	10	11
3·5	·5441	·5453	·5465	·5478	·5490	·5502	·5514	·5527	·5539	·5551	1	2	4	5	6	7	9	10	11
3·6	·5563	·5575	·5587	·5599	·5611	·5623	·5635	·5647	·5658	·5670	1	2	4	5	6	7	8	10	11
3·7	·5682	·5694	·5705	·5717	·5729	·5740	·5752	·5763	·5775	·5786	1	2	3	5	6	7	8	9	10
3·8	·5798	·5809	·5821	·5832	·5843	·5855	·5866	·5877	·5888	·5899	1	2	3	5	6	7	8	9	10
3·9	·5911	·5922	·5933	·5944	·5955	·5966	·5977	·5988	·5999	·6010	1	2	3	4	5	7	8	9	10
4·0	·6021	·6031	·6042	·6053	·6064	·6075	·6085	·6096	·6107	·6117	1	2	3	4	5	6	8	9	10
4·1	·6128	·6138	·6149	·6160	·6170	·6180	·6191	·6201	·6212	·6222	1	2	3	4	5	6	7	8	9
4·2	·6232	·6243	·6253	·6263	·6274	·6284	·6294	·6304	·6314	·6325	1	2	3	4	5	6	7	8	9
4·3	·6335	·6345	·6355	·6365	·6375	·6385	·6395	·6405	·6415	·6425	1	2	3	4	5	6	7	8	9
4·4	·6435	·6444	·6454	·6464	·6474	·6484	·6493	·6503	·6513	·6522	1	2	3	4	5	6	7	8	9
4·5	·6532	·6542	·6551	·6561	·6571	·6580	·6590	·6599	·6609	·6618	1	2	3	4	5	6	7	8	9
4·6	·6628	·6637	·6646	·6656	·6665	·6675	·6684	·6693	·6702	·6712	1	2	3	4	5	6	7	7	8
4·7	·6721	·6730	·6739	·6749	·6758	·6767	·6776	·6785	·6794	·6803	1	2	3	4	5	5	6	7	8
4·8	·6812	·6821	·6830	·6839	·6848	·6857	·6866	·6875	·6884	·6893	1	2	3	4	4	5	6	7	8
4·9	·6902	·6911	·6920	·6928	·6937	·6946	·6955	·6964	·6972	·6981	1	2	3	4	4	5	6	7	8
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5·3	·7243	·7251	·7259	·7267	·7275	·7284	·7292	·7300	·7308	·7316	1	2	2	3	4	5	6	6	7
5·4	·7324	·7332	·7340	·7348	·7356	·7364	·7372	·7380	·7388	·7396	1	2	2	3	4	5	6	6	7

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
5·5	·7404	·7412	·7419	·7427	·7435	·7443	·7451	·7459	·7466	·7474	1	2	2	3	4	5	5	6	7
5·6	·7482	·7490	·7497	·7505	·7513	·7520	·7528	·7536	·7543	·7551	1	2	2	3	4	5	5	6	7
5·7	·7559	·7566	·7574	·7582	·7589	·7597	·7604	·7612	·7619	·7627	1	2	2	3	4	5	5	6	7
5·8	·7634	·7642	·7649	·7657	·7664	·7672	·7679	·7686	·7694	·7701	1	1	2	3	4	4	5	6	7
5·9	·7709	·7716	·7723	·7731	·7738	·7745	·7752	·7760	·7767	·7774	1	1	2	3	4	4	5	6	7
6·0	·7782	·7789	·7796	·7803	·7810	·7818	·7825	·7832	·7839	·7846	1	1	2	3	4	4	5	6	6
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6·6	·8195	·8202	·8209	·8215	·8222	·8228	·8235	·8241	·8248	·8254	1	1	2	3	3	4	5	5	6
6·7	·8261	·8267	·8274	·8280	·8287	·8293	·8299	·8306	·8312	·8319	1	1	2	3	3	4	5	5	6
6·8	·8325	·8331	·8338	·8344	·8351	·8357	·8363	·8370	·8376	·8382	1	1	2	3	3	4	4	5	6
6·9	·8388	·8395	·8401	·8407	·8414	·8420	·8426	·8432	·8439	·8445	1	1	2	2	3	4	4	5	6
7·0	·8451	·8457	·8463	·8470	·8476	·8482	·8488	·8494	·8500	·8506	1	1	2	2	3	4	4	5	6
7·1	·8513	·8519	·8525	·8531	·8537	·8543	·8549	·8555	·8561	·8567	1	1	2	2	3	4	4	5	5
7·2	·8573	·8579	·8585	·8591	·8597	·8603	·8609	·8615	·8621	·8627	1	1	2	2	3	4	4	5	5
7·3	·8633	·8639	·8645	·8651	·8657	·8663	·8669	·8675	·8681	·8686	1	1	2	2	3	4	4	5	5
7·4	·8692	·8698	·8704	·8710	·8716	·8722	·8727	·8733	·8739	·8745	1	1	2	2	3	4	4	5	5
7·5	·8751	·8756	·8762	·8768	·8774	·8779	·8785	·8791	·8797	·8802	1	1	2	2	3	3	4	5	5
7·6	·8808	·8814	·8820	·8825	·8831	·8837	·8842	·8848	·8854	·8859	1	1	2	2	3	3	4	5	5
7·7	·8865	·8871	·8876	·8882	·8887	·8893	·8899	·8904	·8910	·8915	1	1	2	2	3	3	4	4	5
7·8	·8921	·8927	·8932	·8938	·8943	·8949	·8954	·8960	·8965	·8971	1	1	2	2	3	3	4	4	5
7·9	·8976	·8982	·8987	·8993	·8998	·9004	·9009	·9015	·9020	·9025	1	1	2	2	3	3	4	4	5
8·0	·9031	·9036	·9042	·9047	·9053	·9058	·9063	·9069	·9074	·9079	1	1	2	2	3	3	4	4	5
8·1	·9085	·9090	·9096	·9101	·9106	·9112	·9117	·9122	·9128	·9133	1	1	2	2	3	3	4	4	5
8·2	·9138	·9143	·9149	·9154	·9159	·9165	·9170	·9175	·9180	·9186	1	1	2	2	3	3	4	4	5
8·3	·9191	·9196	·9201	·9206	·9212	·9217	·9222	·9227	·9232	·9238	1	1	2	2	3	3	4	4	5
8·4	·9243	·9248	·9253	·9258	·9263	·9269	·9274	·9279	·9284	·9289	1	1	2	2	3	3	4	4	5
8·5	·9294	·9299	·9304	·9309	·9315	·9320	·9325	·9330	·9335	·9340	1	1	2	2	3	3	4	4	5
8·6	·9345	·9350	·9355	·9360	·9365	·9370	·9375	·9380	·9385	·9390	1	1	2	2	3	3	4	4	5
8·7	·9395	·9400	·9405	·9410	·9415	·9420	·9425	·9430	·9435	·9440	0	1	1	2	2	3	3	4	4
8·8	·9445	·9450	·9455	·9460	·9465	·9469	·9474	·9479	·9484	·9489	0	1	1	2	2	3	3	4	4
8·9	·9494	·9499	·9504	·9509	·9513	·9518	·9523	·9528	·9533	·9538	0	1	1	2	2	3	3	4	4
9·0	·9542	·9547	·9552	·9557	·9562	·9566	·9571	·9576	·9581	·9586	0	1	1	2	2	3	3	4	4
9·1	·9590	·9595	·9600	·9605	·9609	·9614	·9619	·9624	·9628	·9633	0	1	1	2	2	3	3	4	4
9·2	·9638	·9643	·9647	·9652	·9657	·9661	·9666	·9671	·9675	·9680	0	1	1	2	2	3	3	4	4
9·3	·9685	·9689	·9694	·9699	·9703	·9708	·9713	·9717	·9722	·9727	0	1	1	2	2	3	3	4	4
9·4	·9731	·9736	·9741	·9745	·9750	·9754	·9759	·9763	·9768	·9773	0	1	1	2	2	3	3	4	4
9·5	·9777	·9782	·9786	·9791	·9795	·9800	·9805	·9809	·9814	·9818	0	1	1	2	2	3	3	4	4
9·6	·9823	·9827	·9832	·9836	·9841	·9845	·9850	·9854	·9859	·9863	0	1	1	2	2	3	3	4	4
9·7	·9868	·9872	·9877	·9881	·9886	·9890	·9894	·9899	·9903	·9908	0	1	1	2	2	3	3	4	4
9·8	·9912	·9917	·9921	·9926	·9930	·9934	·9939	·9943	·9948	·9952	0	1	1	2	2	3	3	4	4
9·9	·9956	·9961	·9965	·9969	·9974	·9978	·9983	·9987	·9991	·9996	0	1	1	2	2	3	3	3	4

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·00	1·000	1·002	1·005	1·007	1·009	1·012	1·014	1·016	1·019	1·021	0	0	1	1	1	1	2	2	2
·01	1·023	1·026	1·028	1·030	1·033	1·035	1·038	1·040	1·042	1·045	0	0	1	1	1	1	2	2	2
·02	1·047	1·050	1·052	1·054	1·057	1·059	1·062	1·064	1·067	1·069	0	0	1	1	1	1	2	2	2
·03	1·072	1·074	1·076	1·079	1·081	1·084	1·086	1·089	1·091	1·094	0	0	1	1	1	1	2	2	2
·04	1·096	1·099	1·102	1·104	1·107	1·109	1·112	1·114	1·117	1·119	0	1	1	1	1	2	2	2	2
·05	1·122	1·125	1·127	1·130	1·132	1·135	1·138	1·140	1·143	1·146	0	1	1	1	1	2	2	2	2
·06	1·148	1·151	1·153	1·156	1·159	1·161	1·164	1·167	1·169	1·172	0	1	1	1	1	2	2	2	2
·07	1·175	1·178	1·180	1·183	1·186	1·189	1·191	1·194	1·197	1·199	0	1	1	1	1	2	2	2	2
·08	1·202	1·205	1·208	1·211	1·213	1·216	1·219	1·222	1·225	1·227	0	1	1	1	1	2	2	2	2
·09	1·230	1·233	1·236	1·239	1·242	1·245	1·247	1·250	1·253	1·256	0	1	1	1	1	2	2	2	3
·10	1·259	1·262	1·265	1·268	1·271	1·274	1·276	1·279	1·282	1·285	0	1	1	1	1	2	2	2	3
·11	1·288	1·291	1·294	1·297	1·300	1·303	1·306	1·309	1·312	1·315	0	1	1	1	2	2	2	2	3
·12	1·318	1·321	1·324	1·327	1·330	1·334	1·337	1·340	1·343	1·346	0	1	1	1	2	2	2	2	3
·13	1·349	1·352	1·355	1·358	1·361	1·365	1·368	1·371	1·374	1·377	0	1	1	1	2	2	2	3	3
·14	1·380	1·384	1·387	1·390	1·393	1·396	1·400	1·403	1·406	1·409	0	1	1	1	2	2	2	3	3
·15	1·413	1·416	1·419	1·422	1·426	1·429	1·432	1·435	1·439	1·442	0	1	1	1	2	2	2	3	3
·16	1·445	1·449	1·452	1·455	1·459	1·462	1·466	1·469	1·472	1·476	0	1	1	1	2	2	2	3	3
·17	1·479	1·483	1·486	1·489	1·493	1·496	1·500	1·503	1·507	1·510	0	1	1	1	2	2	2	3	3
·18	1·514	1·517	1·521	1·524	1·528	1·531	1·535	1·538	1·542	1·545	0	1	1	1	2	2	2	3	3
·19	1·549	1·552	1·556	1·560	1·563	1·567	1·570	1·574	1·578	1·581	0	1	1	1	2	2	3	3	3
·20	1·585	1·589	1·592	1·596	1·600	1·603	1·607	1·611	1·614	1·618	0	1	1	1	2	2	3	3	3
·21	1·622	1·626	1·629	1·633	1·637	1·641	1·644	1·648	1·652	1·656	0	1	1	2	2	2	3	3	3
·22	1·660	1·663	1·667	1·671	1·675	1·679	1·683	1·687	1·690	1·694	0	1	1	2	2	2	3	3	3
·23	1·698	1·702	1·706	1·710	1·714	1·718	1·722	1·726	1·730	1·734	0	1	1	2	2	2	3	3	4
·24	1·738	1·742	1·746	1·750	1·754	1·758	1·762	1·766	1·770	1·774	0	1	1	2	2	2	3	3	4
·25	1·778	1·782	1·786	1·791	1·795	1·799	1·803	1·807	1·811	1·816	0	1	1	2	2	2	3	3	4
·26	1·820	1·824	1·828	1·832	1·837	1·841	1·845	1·849	1·854	1·858	0	1	1	2	2	3	3	3	4
·27	1·862	1·866	1·871	1·875	1·879	1·884	1·888	1·892	1·897	1·901	0	1	1	2	2	3	3	3	4
·28	1·905	1·910	1·914	1·919	1·923	1·928	1·932	1·936	1·941	1·945	0	1	1	2	2	3	3	4	4
·29	1·950	1·954	1·959	1·963	1·968	1·972	1·977	1·982	1·986	1·991	0	1	1	2	2	3	3	4	4
·30	1·995	2·000	2·004	2·009	2·014	2·018	2·023	2·028	2·032	2·037	0	1	1	2	2	3	3	4	4
·31	2·042	2·046	2·051	2·056	2·061	2·065	2·070	2·075	2·080	2·084	0	1	1	2	2	3	3	4	4
·32	2·089	2·094	2·099	2·104	2·109	2·113	2·118	2·123	2·128	2·133	0	1	1	2	2	3	3	4	4
·33	2·138	2·143	2·148	2·153	2·158	2·163	2·168	2·173	2·178	2·183	0	1	1	2	2	3	3	4	4
·34	2·188	2·193	2·198	2·203	2·208	2·213	2·218	2·223	2·228	2·234	1	1	2	2	3	3	4	4	5
·35	2·239	2·244	2·249	2·254	2·259	2·265	2·270	2·275	2·280	2·286	1	1	2	2	3	3	4	4	5
·36	2·291	2·296	2·301	2·307	2·312	2·317	2·323	2·328	2·333	2·339	1	1	2	2	3	3	4	4	5
·37	2·344	2·350	2·355	2·360	2·366	2·371	2·377	2·382	2·388	2·393	1	1	2	2	3	3	4	4	5
·38	2·399	2·404	2·410	2·415	2·421	2·427	2·432	2·438	2·443	2·449	1	1	2	2	3	3	4	4	5
·39	2·455	2·460	2·466	2·472	2·477	2·483	2·489	2·495	2·500	2·506	1	1	2	2	3	3	4	4	5
·40	2·512	2·518	2·523	2·529	2·535	2·541	2·547	2·553	2·559	2·564	1	1	2	2	3	4	4	5	5
·41	2·570	2·576	2·582	2·588	2·594	2·600	2·606	2·612	2·618	2·624	1	1	2	2	3	4	4	5	5
·42	2·630	2·636	2·642	2·649	2·655	2·661	2·667	2·673	2·679	2·685	1	1	2	2	3	4	4	5	6
·43	2·692	2·698	2·704	2·710	2·716	2·723	2·729	2·735	2·742	2·748	1	1	2	2	3	4	4	5	6
·44	2·754	2·761	2·767	2·773	2·780	2·786	2·793	2·799	2·805	2·812	1	1	2	3	3	4	4	5	6
·45	2·818	2·825	2·831	2·838	2·844	2·851	2·858	2·864	2·871	2·877	1	1	2	3	3	4	5	5	6
·46	2·884	2·891	2·897	2·904	2·911	2·917	2·924	2·931	2·938	2·944	1	1	2	3	3	4	5	5	6
·47	2·951	2·958	2·965	2·972	2·979	2·985	2·992	2·999	3·006	3·013	1	1	2	3	3	4	5	5	6
·48	3·020	3·027	3·034	3·041	3·048	3·055	3·062	3·069	3·076	3·083	1	1	2	3	4	4	5	6	6
·49	3·090	3·097	3·105	3·112	3·119	3·126	3·133	3·141	3·148	3·155	1	1	2	3	4	4	5	6	6

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·50	3·162	3·170	3·177	3·184	3·192	3·199	3·206	3·214	3·221	3·228	1	1	2	3	4	4	5	6	7
·51	3·236	3·243	3·251	3·258	3·266	3·273	3·281	3·289	3·296	3·304	1	2	2	3	4	5	5	6	7
·52	3·311	3·319	3·327	3·334	3·342	3·350	3·357	3·365	3·373	3·381	1	2	2	3	4	5	5	6	7
·53	3·388	3·396	3·404	3·412	3·420	3·428	3·436	3·443	3·451	3·459	1	2	2	3	4	5	6	6	7
·54	3·467	3·475	3·483	3·491	3·499	3·508	3·516	3·524	3·532	3·540	1	2	2	3	4	5	6	6	7
·55	3·548	3·556	3·565	3·573	3·581	3·589	3·597	3·606	3·614	3·622	1	2	2	3	4	5	6	7	7
·56	3·631	3·639	3·648	3·656	3·664	3·673	3·681	3·690	3·698	3·707	1	2	3	3	4	5	6	7	8
·57	3·715	3·724	3·733	3·741	3·750	3·758	3·767	3·776	3·784	3·793	1	2	3	3	4	5	6	7	8
·58	3·802	3·811	3·819	3·828	3·837	3·846	3·855	3·864	3·873	3·882	1	2	3	4	4	5	6	7	8
·59	3·890	3·899	3·908	3·917	3·926	3·936	3·945	3·954	3·963	3·972	1	2	3	4	5	5	6	7	8
·60	3·981	3·990	3·999	4·009	4·018	4·027	4·036	4·046	4·055	4·064	1	2	3	4	5	6	6	7	8
·61	4·074	4·083	4·093	4·102	4·111	4·121	4·130	4·140	4·150	4·159	1	2	3	4	5	6	7	8	9
·62	4·169	4·178	4·188	4·198	4·207	4·217	4·227	4·236	4·246	4·256	1	2	3	4	5	6	7	8	9
·63	4·266	4·276	4·285	4·295	4·305	4·315	4·325	4·335	4·345	4·355	1	2	3	4	5	6	7	8	9
·64	4·365	4·375	4·385	4·395	4·406	4·416	4·426	4·436	4·446	4·457	1	2	3	4	5	6	7	8	9
·65	4·467	4·477	4·487	4·498	4·508	4·519	4·529	4·539	4·550	4·560	1	2	3	4	5	6	7	8	9
·66	4·571	4·581	4·592	4·603	4·613	4·624	4·634	4·645	4·656	4·667	1	2	3	4	5	6	7	9	10
·67	4·677	4·688	4·699	4·710	4·721	4·732	4·742	4·753	4·764	4·775	1	2	3	4	5	7	8	9	10
·68	4·786	4·797	4·808	4·819	4·831	4·842	4·853	4·864	4·875	4·887	1	2	3	4	6	7	8	9	10
·69	4·898	4·909	4·920	4·932	4·943	4·955	4·966	4·977	4·989	5·000	1	2	3	5	6	7	8	9	10
·70	5·012	5·023	5·035	5·047	5·058	5·070	5·082	5·093	5·105	5·117	1	2	4	5	6	7	8	9	11
·71	5·129	5·140	5·152	5·164	5·176	5·188	5·200	5·212	5·224	5·236	1	2	4	5	6	7	8	10	11
·72	5·248	5·260	5·272	5·284	5·297	5·309	5·321	5·338	5·346	5·358	1	2	4	5	6	7	9	10	11
·73	5·370	5·383	5·395	5·408	5·420	5·433	5·445	5·458	5·470	5·483	1	3	4	5	6	8	9	10	11
·74	5·495	5·508	5·521	5·534	5·546	5·559	5·572	5·585	5·598	5·610	1	3	4	5	6	8	9	10	12
·75	5·623	5·636	5·649	5·662	5·675	5·689	5·702	5·715	5·728	5·741	1	3	4	5	7	8	9	10	12
·76	5·754	5·768	5·781	5·794	5·808	5·821	5·834	5·848	5·861	5·875	1	3	4	5	7	8	9	11	12
·77	5·888	5·902	5·916	5·929	5·943	5·957	5·970	5·984	5·998	6·012	1	3	4	5	7	8	10	11	12
·78	6·026	6·039	6·053	6·067	6·081	6·095	6·109	6·124	6·138	6·152	1	3	4	6	7	8	10	11	13
·79	6·166	6·180	6·194	6·209	6·223	6·237	6·252	6·266	6·281	6·295	1	3	4	6	7	9	10	11	13
·80	6·310	6·324	6·339	6·353	6·368	6·383	6·397	6·412	6·427	6·442	1	3	4	6	7	9	10	12	13
·81	6·457	6·471	6·486	6·501	6·516	6·531	6·546	6·561	6·577	6·592	2	3	5	6	8	9	11	12	14
·82	6·607	6·622	6·637	6·653	6·668	6·683	6·699	6·714	6·730	6·745	2	3	5	6	8	9	11	12	14
·83	6·761	6·776	6·792	6·808	6·823	6·839	6·855	6·871	6·887	6·902	2	3	5	6	8	9	11	13	14
·84	6·918	6·934	6·950	6·966	6·982	6·998	7·015	7·031	7·047	7·063	2	3	5	6	8	10	11	13	15
·85	7·079	7·096	7·112	7·129	7·145	7·161	7·178	7·194	7·211	7·228	2	3	5	7	8	10	12	13	15
·86	7·244	7·261	7·278	7·295	7·311	7·328	7·345	7·362	7·379	7·396	2	3	5	7	8	10	12	13	15
·87	7·413	7·430	7·447	7·464	7·482	7·499	7·516	7·534	7·551	7·568	2	3	5	7	9	10	12	14	16
·88	7·586	7·603	7·621	7·638	7·656	7·674	7·691	7·709	7·727	7·745	2	4	5	7	9	11	12	14	16
·89	7·762	7·730	7·798	7·816	7·834	7·852	7·870	7·889	7·907	7·925	2	4	5	7	9	11	13	14	16
·90	7·943	7·962	7·980	7·998	8·017	8·035	8·054	8·072	8·091	8·110	2	4	6	7	9	11	13	15	17
·91	8·128	8·147	8·166	8·185	8·204	8·222	8·241	8·260	8·279	8·299	2	4	6	8	9	11	13	15	17
·92	8·318	8·337	8·356	8·375	8·395	8·414	8·433	8·453	8·472	8·492	2	4	6	8	10	12	14	15	17
·93	8·511	8·531	8·551	8·570	8·590	8·610	8·630	8·650	8·670	8·690	2	4	6	8	10	12	14	16	18
·94	8·710	8·730	8·750	8·770	8·790	8·810	8·831	8·851	8·872	8·892	2	4	6	8	10	12	14	16	18
·95	8·913	8·954	8·954	8·974	8·995	9·016	9·036	9·057	9·078	9·099	2	4	6	8	10	12	15	17	19
·96	9·120	9·141	9·162	9·183	9·204	9·226	9·247	9·268	9·290	9·311	2	4	6	8	11	13	15	17	19
·97	9·333	9·354	9·376	9·397	9·419	9·441	9·462	9·484	9·506	9·528	2	4	7	9	11	13	15	17	20
·98	9·550	9·572	9·594	9·616	9·638	9·661	9·683	9·705	9·727	9·750	2	4	7	9	11	13	16	18	20
·99	9·772	9·795	9·817	9·840	9·863	9·886	9·908	9·931	9·954	9·977	2	5	7	9	11	14	16	18	20

Trigonometrical Ratios

A (In degrees)	sin A	cos A	tan A	A (In degrees)	sin A	cos A	tan A
0	·0000	1·000	·0000	45	·7071	·7071	1·0000
1	·0175	·9998	·0175	46	·7193	·6947	1·0355
2	·0349	·9994	·0349	47	·7314	·6820	1·0724
3	·0523	·9986	·0524	48	·7431	·6691	1·1106
4	·0698	·9976	·0699	49	·7547	·6561	1·1504
5	·0872	·9962	·0875	50	·7660	·6428	1·1918
6	·1045	·9945	·1051	51	·7771	·6293	1·2349
7	·1219	·9925	·1228	52	·7880	·6157	1·2799
8	·1392	·9903	·1405	53	·7986	·6018	1·3270
9	·1564	·9877	·1584	54	·8090	·5878	1·3764
10	·1736	·9848	·1763	55	·8192	·5736	1·4281
11	·1908	·9816	·1944	56	·8290	·5592	1·4826
12	·2079	·9781	·2126	57	·8387	·5446	1·5399
13	·2250	·9744	·2309	58	·8480	·5299	1·6003
14	·2419	·9703	·2493	59	·8572	·5150	1·6643
15	·2588	·9659	·2679	60	·8660	·5000	1·7321
16	·2756	·9613	·2867	61	·8746	·4848	1·8040
17	·2924	·9563	·3057	62	·8829	·4695	1·8807
18	·3090	·9511	·3249	63	·8910	·4540	1·9626
19	·3256	·9455	·3443	64	·8988	·4384	2·0[illegible]3
20	·3420	·9397	·3640	65	·9063	·4226	2·1[illegible]5
21	·3584	·9336	·3839	66	·9135	·4067	2·2460
22	·3746	·9272	·4040	67	·9205	·3907	2·3559
23	·3907	·9205	·4245	68	·9272	·3746	2·4751
24	·4067	·9135	·4452	69	·9336	·3584	2·6051
25	·4226	·9063	·4663	70	·9397	·3420	2·7475
26	·4384	·8988	·4877	71	·9455	·3256	2·9042
27	·4540	·8910	·5095	72	·9511	·3090	3·0777
28	·4695	·8829	·5317	73	·9563	·2924	3·2709
29	·4848	·8746	·5543	74	·9613	·2756	3·4874
30	·5000	·8660	·5774	75	·9659	·2588	3·7321
31	·5150	·8572	·6009	76	·9703	·2419	4·0108
32	·5299	·8480	·6249	77	·9744	·2250	4·3315
33	·5446	·8387	·6494	78	·9781	·2079	4·7046
34	·5592	·8290	·6745	79	·9816	·1908	5·1446
35	·5736	·8192	·7002	80	·9848	·1736	5·6713
36	·5878	·8090	·7265	81	·9877	·1564	6·3138
37	·6018	·7986	·7536	82	·9903	·1392	7·1154
38	·6157	·7880	·7813	83	·9925	·1219	8·1443
39	·6293	·7771	·8098	84	·9945	·1045	9·514
40	·6428	·7660	·8391	85	·9962	·0872	11·43
41	·6561	·7547	·8693	86	·9976	·0698	14·30
42	·6691	·7431	·9004	87	·9986	·0523	19·08
43	·6820	·7314	·9325	88	·9994	·0349	28·64
44	·6947	·7193	·9657	89	·9998	·0175	57·29

Squares, Cubes, Square roots and cube roots

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1	1	51	2601	132651	7·141	3·708
2	4	8	1·414	1·260	52	2704	140608	7·211	3·733
3	9	27	1·732	1·442	53	2809	148877	7·280	3·756
4	16	64	2·000	1·587	54	2916	157464	7·348	3·780
5	25	125	2·236	1·710	55	3025	166375	7·416	3·808
6	36	216	2·449	1·817	56	3136	175616	7·483	3·826
7	49	343	2·646	1·913	57	3249	185193	7·550	3·849
8	64	512	2·828	2·000	58	3364	195112	7·616	3·871
9	81	729	3·000	2·080	59	3481	205379	7·681	3·893
10	100	1000	3·162	2·154	60	3600	216000	7·746	3·915
11	121	1331	3·317	2·224	61	3721	226981	7·810	3·936
12	144	1728	3·463	2·289	62	3844	238328	7·874	3·958
13	169	2197	3·606	2·351	63	3969	250047	7·937	3·979
14	196	2744	3·742	2·410	64	4096	262144	8·000	4·000
15	225	3375	3·873	2·466	65	4225	274625	8·062	4·021
16	256	4096	4·000	2·520	66	4356	287496	8·124	4·041
17	289	4913	4·123	2·571	67	4489	300763	8·185	4·062
18	324	5832	4·243	2·621	68	4624	314432	8·246	4·082
19	361	6859	4·359	2·668	69	4761	328509	8·307	4·102
20	400	8000	4·472	2·714	70	4900	343000	8·367	4·121
21	441	9261	4·583	2·759	71	5041	357911	8·426	4·141
22	484	10648	4·690	2·802	72	5184	373248	8·485	4·160
23	529	12167	4·796	2·844	73	5329	389017	8·544	4·179
24	576	13824	4·899	2·884	74	5476	405224	8·602	4·198
25	625	15625	5·000	2·924	75	5625	421875	8·660	4·217
26	676	17576	5·099	2·962	76	5776	438976	8·718	4·236
27	729	19683	5·196	3·000	77	5929	456533	8·775	4·254
28	784	21952	5·291	3·307	78	6084	474552	8·832	4·273
29	841	24389	5·385	3·072	79	6241	493039	8·888	4·291
30	900	27000	5·477	3·107	80	6400	512000	8·944	4·309
31	961	29791	5·568	3·141	81	6561	531441	9·000	4·327
32	1024	32768	5·657	3·175	82	6724	551368	9·055	4·344
33	1089	35937	5·745	3·208	83	6889	571787	9·110	4·362
34	1156	39304	5·831	3·240	84	7056	592704	9·165	4·380
35	1225	42875	5·916	3·271	85	7225	614125	9·220	4·397
36	1296	46656	6·000	3·302	86	7396	636056	9·274	4·414
37	1369	50653	6·083	3·332	87	7569	658503	9·327	4·431
38	1444	54872	6·164	3·362	88	7744	681472	9·381	4·448
39	1521	59319	6·245	3·391	89	7921	704969	9·434	4·465
40	1600	64000	6·325	3·420	90	8100	729000	9·487	4·481
41	1681	88921	6·403	3·448	91	8281	753571	9·539	4·493
42	1764	74088	6·481	3·476	92	8464	778688	9·592	4·514
43	1849	79507	6·557	3·503	93	8649	804357	9·644	4·531
44	1936	85184	6·633	3·530	94	8836	830584	9·695	4·547
45	2025	91125	6·703	3·557	95	9025	857375	9·747	4·563
46	2116	97336	6·782	3·583	96	9216	884736	9·798	4·579
47	2209	103823	6·866	3·609	97	9409	912673	9·849	4·595
48	2304	110592	6·928	3·634	98	9604	941192	9·899	4·610
49	2401	117649	7·000	3·659	99	9801	970299	9·950	4·626
50	2500	125000	7·071	3·684	100	10000	1000000	10·000	4·642

Reciprocals

	0	1	2	3	4	5	6	7	8	9	Mean Difference (Subtract) 1	2	3	4	5	6	7	8	9
1·0	1·000	·9901	·9804	·9709	·9615	·9524	·9434	·9346	·9259	·9174	9	18	28	37	46	55	64	74	83
1·1	·9091	·9009	·8929	·8850	·8772	·8696	·8621	·8547	·8475	·8403	8	15	23	31	38	46	53	61	69
1·2	·8333	·8264	·8197	·8130	·8065	·8000	·7937	·7874	·7813	·7752	7	13	20	26	33	39	46	52	59
1·3	·7692	·7634	·7576	·7519	·7463	·7407	·7353	·7299	·7246	·7194	6	11	17	22	28	33	39	44	50
1·4	·7143	·7092	·7042	·6993	·6944	·6897	·6849	·6803	·6757	·6711	5	10	14	19	24	29	33	38	43
1·5	·6667	·6623	·6579	·6536	·6494	·6452	·6410	·6369	·6329	·6289	4	8	13	17	21	25	29	34	38
1·6	·6250	·6211	·6173	·6135	·6098	·6061	·6024	·5988	·5952	·5917	4	7	11	15	19	22	26	29	33
1·7	·5882	·5848	·5814	·5780	·5747	·5714	·5682	·5650	·5618	·5587	3	7	10	13	16	20	23	26	30
1·8	·5556	·5525	·5495	·5464	·5435	·5405	·5376	·5348	·5319	·5291	3	6	9	12	15	17	20	23	26
1·9	·5263	·5236	·5208	·5181	·5155	·5128	·5102	·5076	·5051	·5025	3	5	8	11	13	16	18	21	24
2·0	·5000	·4975	·4950	·4926	·4902	·4878	·4854	·4831	·4808	·4785	2	5	7	10	12	14	17	19	22
2·1	·4762	·4739	·4717	·4695	·4673	·4651	·4630	·4608	·4587	·4566	2	4	7	9	11	13	15	17	20
2·2	·4545	·4525	·4505	·4484	·4464	·4444	·4425	·4405	·4386	·4367	2	4	6	8	10	12	14	16	18
2·3	·4348	·4329	·4310	·4292	·4274	·4255	·4237	·4219	·4202	·4184	2	4	5	7	9	11	13	14	16
2·4	·4167	·4149	·4132	·4115	·4098	·4082	·4065	·4049	·4032	·4016	2	3	5	7	8	10	12	13	15
2·5	·4000	·3984	·3968	·3953	·3937	·3922	·3906	·3891	·3876	·3861	2	3	5	6	8	9	11	12	14
2·6	·3846	·3831	·3817	·3802	·3788	·3774	·3759	·3745	·3731	·3717	1	3	4	6	7	8	10	11	13
2·7	·3704	·3690	·3676	·3663	·3650	·3636	·3623	·3610	·3597	·3584	1	3	4	5	7	8	9	11	12
2·8	·3571	·3559	·3546	·3534	·3521	·3509	·3497	·3484	·3472	·3460	1	2	4	5	6	7	9	10	11
2·9	·3448	·3436	·3425	·3413	·3401	·3390	·3378	·3367	·3356	·3344	1	2	3	5	6	7	8	9	10
3·0	·3333	·3322	·3311	·3300	·3289	·3279	·3268	·3257	·3247	·3236	1	2	3	4	5	6	7	9	10
3·1	·3226	·3215	·3205	·3195	·3185	·3175	·3165	·3155	·3145	·3135	1	2	3	4	5	6	7	8	9
3·2	·3125	·3115	·3106	·3096	·3086	·3077	·3067	·3058	·3049	·3040	1	2	3	4	5	6	7	8	9
3·3	·3030	·3021	·3012	·3003	·2994	·2985	·2976	·2967	·2959	·2950	1	2	3	4	4	5	6	7	8
3·4	·2941	·2933	·2924	·2915	·2907	·2899	·2890	·2882	·2874	·2865	1	2	3	3	4	5	6	7	8
3·5	·2857	·2849	·2841	·2833	·2825	·2817	·2809	·2801	·2793	·2786	1	2	2	3	4	5	6	6	7
3·6	·2778	·2770	·2762	·2755	·2747	·2740	·2732	·2725	·2717	·2710	1	2	2	3	4	5	5	6	7
3·7	·2703	·2695	·2688	·2681	·2674	·2667	·2660	·2653	·2646	·2639	1	1	2	3	4	4	5	6	6
3·8	·2632	·2625	·2618	·2611	·2604	·2597	·2591	·2584	·2577	·2571	1	1	2	3	3	4	5	5	6
3·9	·2564	·2558	·2551	·2545	·2538	·2532	·2525	·2519	·2513	·2506	1	1	2	3	3	4	4	5	6
4·0	·2500	·2494	·2488	·2481	·2475	·2469	·2463	·2457	·2451	·2445	1	1	2	2	3	4	4	5	5
4·1	·2439	·2433	·2427	·2421	·2415	·2410	·2404	·2398	·2392	·2387	1	1	2	2	3	3	4	5	5
4·2	·2381	·2375	·2370	·2364	·2358	·2353	·2347	·2342	·2336	·2331	1	1	2	2	3	3	4	4	5
4·3	·2326	·2320	·2315	·2309	·2304	·2299	·2294	·2288	·2283	·2278	1	1	2	2	3	3	4	4	5
4·4	·2273	·2268	·2262	·2257	·2252	·2247	·2242	·2237	·2232	·2227	1	1	2	2	3	3	4	4	5
4·5	·2222	·2217	·2212	·2208	·2203	·2198	·2193	·2188	·2183	·2179	0	1	1	2	2	3	3	4	4
4·6	·2174	·2169	·2165	·2160	·2155	·2151	·2146	·2141	·2137	·2132	0	1	1	2	2	3	3	4	4
4·7	·2128	·2123	·2119	·2114	·2110	·2105	·2101	·2096	·2092	·2088	0	1	1	2	2	3	3	4	4
4·8	·2083	·2079	·2075	·2070	·2066	·2062	·2058	·2053	·2049	·2045	0	1	1	2	2	3	3	3	4
4·9	·2041	·2037	·2033	·2028	·2024	·2020	·2016	·2012	·2008	·2004	0	1	1	2	2	2	3	3	4
5·0	·2000	·1996	·1992	·1988	·1984	·1980	·1976	·1972	·1969	·1965	0	1	1	2	2	2	3	3	4
5·1	·1961	·1957	·1953	·1949	·1946	·1942	·1938	·1934	·1931	·1927	0	1	1	2	2	2	3	3	3
5·2	·1923	·1919	·1916	·1912	·1908	·1905	·1901	·1898	·1894	·1890	0	1	1	1	2	2	3	3	3
5·3	·1887	·1883	·1880	·1876	·1873	·1869	·1866	·1862	·1859	·1855	0	1	1	1	2	2	2	3	3
5·4	·1852	·1848	·1845	·1842	·1838	·1835	·1832	·1828	·1825	·1821	0	1	1	1	2	2	2	3	3

Reciprocals

	0	1	2	3	4	5	6	7	8	9	Mean Difference (Subtract) 1	2	3	4	5	6	7	8	9
5·5	·1818	·1815	·1812	·1808	·1805	·1802	·1799	·1795	·1792	·1739	0	1	1	1	2	2	2	3	3
5·6	·1786	·1783	·1779	·1776	·1773	·1770	·1767	·1764	·1761	·1757	0	1	1	1	2	2	2	3	3
5·7	·1754	·1751	·1748	·1745	·1742	·1739	·1736	·1733	·1730	·1727	0	1	1	1	1	2	2	2	3
5·8	·1724	·1721	·1718	·1715	·1712	·1709	·1706	·1704	·1701	·1698	0	1	1	1	1	2	2	2	3
5·9	·1695	·1692	·1689	·1686	·1684	·1681	·1678	·1675	·1672	·1669	0	1	1	1	1	2	2	2	3
6·0	·1667	·1664	·1661	·1658	·1656	·1653	·1650	·1647	·1645	·1642	0	1	1	1	1	2	2	2	3
6·1	·1639	·1637	·1634	·1631	·1629	·1626	·1623	·1621	·1618	·1616	0	1	1	1	1	2	2	2	2
6·2	·1613	·1610	·1608	·1605	·1603	·1600	·1597	·1595	·1592	·1590	0	1	1	1	1	2	2	2	2
6·3	·1587	·1585	·1582	·1580	·1577	·1575	·1572	·1570	·1567	·1565	0	0	1	1	1	1	2	2	2
6·4	·1562	·1560	·1558	·1555	·1553	·1550	·1548	·1546	·1543	·1541	0	0	1	1	1	1	2	2	2
6·5	·1538	·1536	·1534	·1531	·1529	·1527	·1524	·1522	·1520	·1517	0	0	1	1	1	1	2	2	2
6·6	·1515	·1513	·1511	·1508	·1506	·1504	·1502	·1499	·1497	·1495	0	0	1	1	1	1	2	2	2
6·7	·1493	·1490	·1488	·1486	·1484	·1481	·1479	·1477	·1475	·1473	0	0	1	1	1	1	2	2	2
6·8	·1471	·1468	·1466	·1464	·1462	·1460	·1458	·1456	·1453	·1451	0	0	1	1	1	1	2	2	2
6·9	·1449	·1447	·1445	·1443	·1441	·1439	·1437	·1435	·1433	·1431	0	0	1	1	1	1	2	2	2
7·0	·1429	·1427	·1425	·1422	·1420	·1418	·1416	·1414	·1412	·1410	0	0	1	1	1	1	1	2	2
7·1	·1408	·1406	·1404	·1403	·1401	·1399	·1397	·1395	·1393	·1391	0	0	1	1	1	1	1	2	2
7·2	·1389	·1387	·1385	·1383	·1381	·1379	·1377	·1376	·1374	·1372	0	0	1	1	1	1	1	2	2
7·3	·1370	·1368	·1366	·1364	·1362	·1361	·1359	·1357	·1355	·1353	0	0	1	1	1	1	1	2	2
7·4	·1351	·1350	·1348	·1346	·1344	·1342	·1340	·1339	·1337	·1335	0	0	1	1	1	1	1	1	2
7·5	·1333	·1332	·1330	·1328	·1326	·1325	·1323	·1321	·1319	·1318	0	0	1	1	1	1	1	1	2
7·6	·1316	·1314	·1312	·1311	·1309	·1307	·1305	·1304	·1302	·1300	0	0	1	1	1	1	1	1	2
7·7	·1299	·1297	·1295	·1294	·1292	·1290	·1289	·1287	·1285	·1284	0	0	0	1	1	1	1	1	1
7·8	·1282	·1280	·1279	·1277	·1276	·1274	·1272	·1271	·1269	·1267	0	0	0	1	1	1	1	1	1
7·9	·1266	·1264	·1263	·1261	·1259	·1258	·1256	·1255	·1253	·1252	0	0	0	1	1	1	1	1	1
8·0	·1250	·1248	·1247	·1245	·1244	·1242	·1241	·1239	·1238	·1236	0	0	0	1	1	1	1	1	1
8·1	·1235	·1233	·1232	·1230	·1229	·1227	·1225	·1224	·1222	·1221	0	0	0	1	1	1	1	1	1
8·2	·1220	·1218	·1217	·1215	·1214	·1212	·1211	·1209	·1208	·1206	0	0	0	1	1	1	1	1	1
8·3	·1205	·1203	·1202	·1200	·1199	·1198	·1196	·1195	·1193	·1192	0	0	0	1	1	1	1	1	1
8·4	·1190	·1189	·1188	·1186	·1185	·1183	·1182	·1181	·1179	·1178	0	0	0	1	1	1	1	1	1
8·5	·1176	·1175	·1174	·1172	·1171	·1170	·1168	·1167	·1166	·1164	0	0	0	1	1	1	1	1	1
8·6	·1163	·1161	·1160	·1159	·1157	·1156	·1155	·1153	·1152	·1151	0	0	0	1	1	1	1	1	1
8·7	·1149	·1148	·1147	·1145	·1144	·1143	·1142	·1140	·1139	·1138	0	0	0	1	1	1	1	1	1
8·8	·1136	·1135	·1134	·1133	·1131	·1130	·1129	·1127	·1126	·1125	0	0	0	1	1	1	1	1	1
8·9	·1124	·1122	·1121	·1120	·1119	·1117	·1116	·1115	·1114	·1112	0	0	0	1	1	1	1	1	1
9·0	·1111	·1110	·1109	·1107	·1106	·1105	·1104	·1103	·1101	·1100	0	0	0	1	1	1	1	1	1
9·1	·1099	·1098	·1096	·1095	·1094	·1093	·1092	·1090	·1089	·1088	0	0	0	0	1	1	1	1	1
9·2	·1087	·1086	·1085	·1083	·1082	·1081	·1080	·1079	·1078	·1076	0	0	0	0	1	1	1	1	1
9·3	·1075	·1074	·1073	·1072	·1071	·1070	·1068	·1067	·1066	·1065	0	0	0	0	1	1	1	1	1
9·4	·1064	·1063	·1062	·1060	·1059	·1058	·1057	·1056	·1055	·1054	0	0	0	0	1	1	1	1	1
9·5	·1053	·1052	·1050	·1049	·1048	·1047	·1046	·1045	·1044	·1043	0	0	0	0	1	1	1	1	1
9·6	·1042	·1041	·1039	·1038	·1037	·1036	·1035	·1034	·1033	·1032	0	0	0	0	1	1	1	1	1
9·7	·1031	·1030	·1029	·1028	·1027	·1026	·1025	·1024	·1022	·1021	0	0	0	0	1	1	1	1	1
9·8	·1020	·1019	·1018	·1017	·1016	·1015	·1014	·1013	·1012	·1011	0	0	0	0	1	1	1	1	1
9·9	·1010	·1009	·1008	·1007	·1006	·1005	·1004	·1003	·1002	·1001	0	0	0	0	0	1	1	1	1

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